FINANCIAL RISK MANAGEMENT

UNIVERSITY OF ANTWERP
FACULTY OF ECONOMICS

Academic year: 2021-2022

Financial Risk Management

an extensive summary

Author:
De Pelsmaker Ward

E-mail address:
w@depelsmaker.be

Contents

Give me liberty, or give me death!

Patrick Henry

1 Introduction

1.1 Introduction to Forward Contracts

- A forward contract is an agreement concluded between two counterparties to buy or sell an underlying asset for a contracted delivery price at a certain point in time in the future.
- We represent a forward contract graphically below. The sale of the underlying asset takes place at time t_0. At this time there is an agreement on the price and the underlying asset/product. The transaction takes place at time T. At time

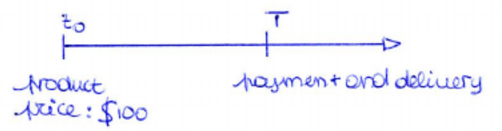


Figure 1.1: The time line of a forward contract.

- Time t_0 is the inception of the forward contract.
  - At this time there is a valid purchase agreement from al legal point of view: there is an agreement upon the price and the product.
  - At this time there is an agreement that the deal will be settled, at some point in the future; however there are no cashflows at this time.
  - The price determined at this time is called the delivery price, this price will be paid at maturity T.
- Time T is called the maturity of the forward contract.
  - At this time, payment and delivery take place.
- Throughout this chapter we will use specific vocabulary and corresponding notations.
  - T = the maturity.
  - $\tau = (T - t_0)$= the time to maturity.
  - DP = the delivery price.
  - F = the forward price.
  - S_{t_0}= the spot price of the underlying at time t_0.

1.2 Parties involved in a forward contract

- There are two parties involved in a forward contract i.e. the long party and the short party.
  1. The long party.
     - The long party has the obligation to buy the underlying.
     - The long will gain from a rising asset price. The forward contract allows him to buy the underlying at a lower price than the spot price at maturity T. This allows him to realize a profit. The long buys the underlying at the delivery price DP via the forward contract and can immediately sell the asset spot at the higher spot price at maturity S_T.
     - In general, the long party is the party in a derivatives contract that profits from a rising price in the underlying.
  2. The short party.
     - The short party has the obligation to sell the underlying.
     - The short will gain from a decreasing asset price. The forward contract allows him to sell the underlying at a higher price than the spot price at maturity T. This allows him to realize a profit. The short buys the underlying spot at the spot price at maturity S_T and immediately sells the asset via the forward contract at the higher delivery price DP.
     - In general, the short party is the party in a derivatives contract that profits from a declining price in the underlying.
- In summary, there is a short party and a long party in a forward contract. The party that is short in the forward contract has the obligation to sell the underlying at maturity T at the contracted delivery price DP. The party that is long in the forward contract has the obligation to buy the underlying at maturity T at the contracted delivery price DP.

1.3 Settlement

- There are two ways in which a forward contract can be settled.
  - Physical delivery of the underlying. The party that is long in the forward contract takes delivery from the party that is short in the contract.
  - Settlement in cash. When a forward contract is settled in cash; the delivery price DP is compared with the the spot price of the underlying at maturity: S_T. This cash amount will switch hands between the 2 counterparties.
- In real life, most derivatives are settled in cash.

1.4 Types of traders in forward contracts

- People trade in forward contracts for a variety of reasons. Based on these reasons we can distinguish different types of traders.

  1. Hedgers. The goal of the hedger is to eliminate cashflow uncertainty from a future transaction. Uncertainty can stem form different sources. Examples are exchange rates, interest rates, commodity prices, etc.
  2. Speculators. The goal of the speculator is to create an exposure to a certain risk.

1.5 Characteristics of forward contracts

- We will now discuss the most important characteristics of the forward contract.

  1. Future contracts are an Over The Counter product and a bilateral contracts. This implies that these contracts are directly negotiated between buyer and seller. For this reason they are highly customizable i.e. the contract can be tailored to the needs of both parties. Because the contract is intuitu personae, the contractual obligations cannot be transferred unilaterally to third party. To get rid of the contract, there are two options:

     a) The investor could seek to find a third party with whom he enters the opposite position. In this case, the original derivative contract remains on the books The exposure to the market risk created by entering into the first forward contract then disappears. However, the exposure to credit risk goes up because now, there are now two contracts and thus two counterparties; both of which can default.

     b) The investor could also try to enter the opposite position with the same counterparty. This process is called ”novation”. This action cancels out the obligations of both parties.

  2. Forward contracts are settled at maturity date. There is only one settlement date.
  3. Both parties are exposed to default risk, this credit risk stems from the time between concluding the agreement and the time of delivery.

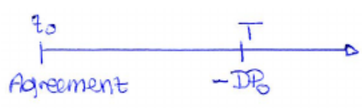


Figure 1.2: The time between agreement and the cashflow, delivery.

4. Forward contract incorporate an element of uncertainty. As we already know, at time t_0 the delivery price DP is agreed upon. However, the cashflow only takes place at maturity T, in return the long receives the underlying with value S_T. The net cashflow (for the long position) becomes $S_T - DP_0$. It is clear that the spot price of the underlying at maturity S_T is the element of uncertainty.

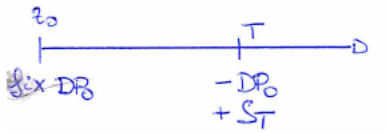


Figure 1.3: The long forward contract and uncertainty.

1.6 The payoff from a future contract

- In general, the payoff from a derivatives contract is denoted by f_T. The payoff from a derivatives contract is the net cashflow generated by the contract at maturity date. It is the value of this derivatives contract at maturity T.
- Forwards are a zero sum game: the payoff from the short position is the negative from the payoff of the long position, mathematically:

$$f_T^{Long\ fw.} = -f_T^{Short\ fw.}$$

- The value of a long position in a forward contract at maturity is the difference between the present value of the underlying asset in the spot market and the delivery price at maturity date. The underlying reasoning is as follows:
  - The long party has the obligation to pay the contracted delivery price DP. In return he acquires the underlying asset. This constitutes a negative cashflow of magnitude DP
  - The long party then has the possibility to immediately sell the underlying asset at the prevailing spot price S_T. This constitutes a positive cashflow of magnitude S_T.
  - The value of a long position in a forward contract is therefore the difference between both cashflows as listed above. Mathematically, this becomes:

$$f_T^{Long\ fw.} = S_T - DP$$

Where:

$f_T^{Long\ fw.}$ = the payoff of a long position in a forward contract, at maturity.
S_T = the spot price of the underlying asset at maturity.
DP = the delivery price.

- The value of a short position in a forward contract is the difference between the present value of the underlying asset in the spot market and the delivery price at maturity date. The underlying reasoning is as follows:
  - The short party has the obligation to sell the underlying asset. In this transaction he receives the contracted delivery price DP. This constitutes a positive cashflow of magnitude DP
  - The cost of selling the underlying asset is equal to the value of that asset at the moment of the transaction. Therefore, the true cost is reflected by looking at the market price of the underlying asset at the time of transaction. The transaction takes place at the maturity T of the forward contract. Therefore, selling the underlying asset constitutes a negative cashflow equal to the spot price of the underlying asset at maturity S_T. Mathematically, this becomes:

    $$f_T^{Short\ fw.} = DP - S_T$$

    Where:

    f_T^{short} = the payoff of a short position in a forward contract, at maturity.

    S_T = the spot price of the underlying asset at maturity.

    DP = the delivery price.

1.7 Payoff diagrams

- We now introduce payoff diagrams. In a payoff diagram, the payoff of a derivatives contract at maturity f_T is displayed as a function of the spot price of the underlying at maturity S_T. The horizontal axis represents the spot price of the underlying value of derivative, at maturity: S_T On the vertical axis, the payoff of the derivative shown, this is the cashflow that is realized at maturity T for a certain spot price S_T of the underlying at maturity The payoff diagram is illustrated graphically below.

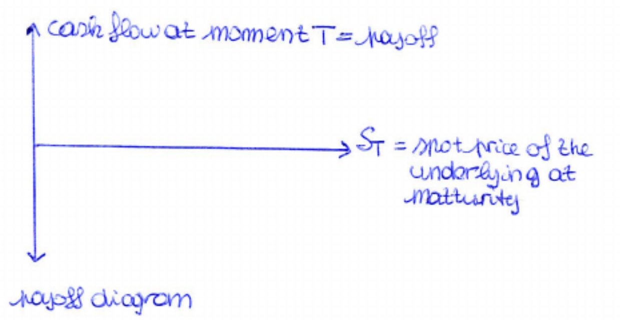


Figure 1.4: A graphical depiction of a payoff diagram.

- Consider the example of a forward contract where the contracted delivery price DP is \$100. The party that is long in the forward contract has the obligation to buy the underlying at a price of \$100 while the party that is short has the obligation to sell the underlying at a price of \$100. We now discuss the payoff diagrams which are depicted in the figure below.
- First, consider the long position in the forward contract.
  - If the spot price of the underlying at maturity is equal to the delivery price, the payoff of the long forward contract is equal to zero. Mathematically:

$$S_T = DP \iff f_T^{Long\ fw.} = 0$$

  - The expression $f_T^{Long\ fw.} = S_T - DP$ is called the payoff function, it gives the value of the long forward contract for the long at maturity T.
  - The payoff of the long forward contract f_T^{long} is a linearly increasing function of the spot price of the underlying at maturity S_T. Possible losses are limited to $-DP$. Possible gains are unlimited.
- Next, consider the short position in the forward contract.
  - If the spot price of the underlying at maturity is equal to the delivery price, the payoff of the short forward contract is equal to zero. Mathematically:

$$S_T = DP \iff f_T^{Short\ fw.} = 0$$

  - The expression $f_T^{Short\ fw.} = DP - S_T$ is called the payoff function, it gives the value of the short forward contract for the long at maturity T.
  - The payoff of the short forward contract f_T^{short} is a linearly decreasing function of the spot price of the underlying at maturity S_T. Possible losses are unlimited. Possible gains are limited to $DP - S_T$.
- Graphically, this becomes:

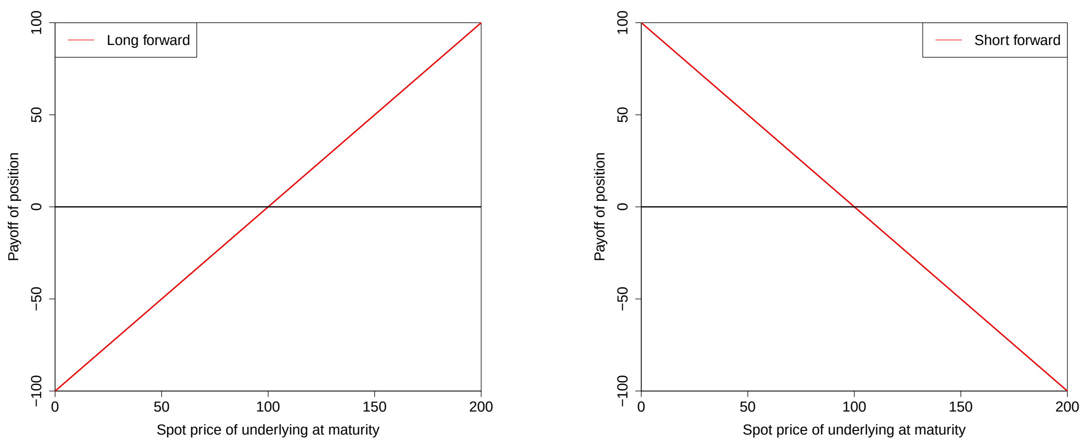


Figure 1.5: Payoff diagram for the long (left) and the short (right) forward contract.

- In summary.

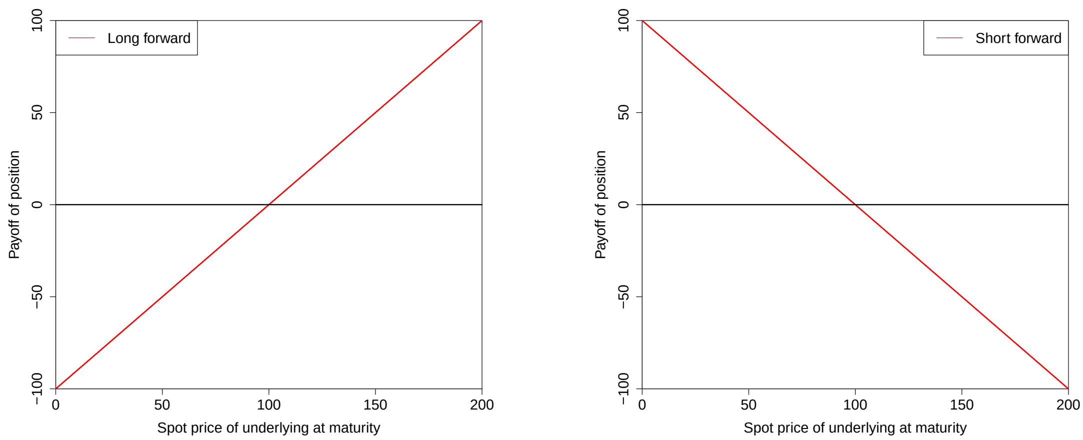


Figure 1.6: Payoff diagram for the long (left) and the short (right) forward contract.

  - The horizontal axis depicts the value of the underlying asset at maturity. The vertical axis depicts the payoff which is the net cashflow at maturity date.
  - When the payoff is equal to zero, the delivery price is equal to the price of the underlying at maturity.
  - The long position gains from rising prices in the underlying. For the long, losses are limited while profits are limitless. This is because the value of the underlying can only drop to zero in the worst case scenario, in which case the long pays the agreed price and receives an asset of no value. In contrast, there is no upper boundary to the payoff because the value of the underlying at maturity can become very high. The long then only pays the contracted delivery price but receives an asset of a much higher value.
  - The short position gains from decreasing prices in the underlying. For the short position losses are limitless and profits are limited. This is because the value of the underlying can only drop to zero which is the best case scenario for the short position. In that case, the short position receives the agreed delivery price for an asset that has no value. In contrast, the value of the underlying can become very high. In that case, the short will always receive the same delivery price but has to deliver the asset which is of much higher value.
  - The range of possible spot prices of the underlying at maturity S_T are given by:

$$S_T \in [0, +\infty]$$

1.8 Determining the delivery price of a forward contract

- At t_0 there are no cashflows, the counter parties are just shaking hands: obligations arise, but nothing else happens; the value of the forward contract, at the time it is initiated is therefore zero.
- For standardized forward contracts, the delivery price is chosen in such a way that the value of the contract upon concluding the agreement is of no value for both parties. Mathematically:

$$f_{t_0}^{long} = f_{t_0}^{short} = 0$$

- When the price of the underlying starts to differ from the spot price at t_0, the forward contract gets value.
- Consider the following example:
  - The value of a painting is $1000.
  - We can take out a loan at 5% p.a.
  - We ask the counter party to buy the painting on the spot market and to sell it forward to us within one year.
  - We now want to determine the forward price.
  - Naturally, the short party expects to receive interest on the spot price of the underlying. He needs to be compensated for postponing his sale. If the short sells the underlying spot, he could invest the proceeds risk free and receive interest.
  - When the forward contract is initialized, it is of no value for both parties. Therefore, neither the short party nor the long party expects an additional return.
  - Therefore, the delivery price is the capitalized amount of the spot price today $FV(S_{t_o})$; calculated using the risk-free interest rate.
  - The delivery price in the example becomes:

$$\$1000 \cdot (1 + 0.05) = \$1050$$

1.9 The bid-ask spread

- The ask price is the price at which we can buy. The bid price is the price at which we can sell. The price for which we can buy i.e. the ask price is always higher then the price at which we can sell i.e. the bid price.
- A bid-ask spread is the amount by which the ask price exceeds the bid price for an asset in the market. The bid-ask spread is essentially the difference between the highest price that a buyer is willing to pay for an asset and the lowest price that a seller is willing to accept. An individual looking to sell will receive the bid price while one looking to buy will pay the ask price. For this reason, there is no such thing as a theoretical forward price in practice; this forward price sits between the bid price and the ask price.
- In what follows, we will ignore the bid-ask spread: we will only consider theoretical prices.

1.10 Arbitrage and the arbitrage-free price

- When the delivery price differs from a theoretical 'forward price', arbitrage opportunities arise.
- Consider the case where the delivery price is greater than the forward price $F > DP$. In this case there exists arbitrage opportunities. An investor can set up an arbitrage portfolio by following the following steps:
  - At time t_0 :
    1. Take out a loan for an amount that equals the spot price of the underlying against the risk free interest rate.
    2. Buy the underlying G
    3. Go short in a forward contract.
  - At maturity date T:
    1. Repay the loan (this amount is the forward price).
    2. Sell the underlying for the delivery price.
- Consider the case where the delivery price is less than the forward price $DP < F$. In this case there exists arbitrage opportunities. An investor can set up an arbitrage portfolio by following the following steps:
  - At time t_0 :
    1. Short the underlying value i.e. sell the underlying value and buy it back on maturity.
    2. Invest this amount (equal to the spot price of the underlying) against the risk free rate.
    3. Take a long position in a futures contract on the underlying value against the lower delivery price.

- At maturity date T:
  1. The loan grants an amount equal to the futures price.
  2. Sale of the underlying value against the lower delivery price.

- We can conclude that the theoretical futures price F is the only arbitrage-free delivery price.
- Consider the example where the arbitrage-free delivery price F is equal \$102. The risk-free interest rate r is equal to 2% and the time to maturity τ is equal to one year.
  - First, suppose Louis wants to enter transactions at \$103 This price is too high because the forward price is equal to \$102. We can therefore set up an arbitrage portfolio by entering a loan and using these funds to enter in a long spot position in the underlying. At the same time we enter a short forward position. At maturity, the short position in the forward contract is settled for the contracted delivery price by delivering the asset that was procured at time t_0. At the same time the loan is settled. Such a portfolio is illustrated in the table below.

Position	t_0	T
Short forward contract	\$0	+\$103
Long spot in underlying	−\$100	
Loan	+\$100	−\$102
Net cashflow	\$0	+\$1

  - Suppose Louis wants to enter transactions at \$101 This price is too high because the forward price is equal to \$102. We can therefore set up an arbitrage portfolio by entering a short position in the underlying, going long in a forward contract and investing the proceeds at the risk-free rate. At maturity, the underlying asset is procured via the long position in the forward contract. The short spot position is closed by returning the asset. The capitalized amount of the investment is also returned. Such a portfolio is illustrated in the table below.

Position	t_0	T
Long forward contract	\$0	−\$101
Short spot in underlying	+\$100	
Investment	−\$100	+\$102
Net cashflow	\$0	+\$1

- We can clearly see that the forward price is the only arbitrage-free delivery price. When the delivery price DP differs from the theoretical forward price F, arbitrage opportunities arise. This means that it is possible to realize a riskless profit by constructing an arbitrage portfolio. Mathematically, this becomes:

$$F = S_{t_0} \cdot (1 + r)$$

1.11 Linear and non-linear products

- Linear products are financial derivatives that have a linear payoff function i.e. the payoff at maturity or the value of the contract at maturity f_T is a linear function of the spot price of the underlying at maturity S_T.
- We can easily see that a forward contract is a linear product: every price increase of the underlying value with \$1 results in a price increase/decrease of the derivative product (the forward contract), with \$1. The payoff profile of the forward contract has the same shape as the payoff profile of the underlying value. Forward contracts are thus a special case of linear products: the slope of the payoff diagram is 45°.
- This means that the payoff profile of a forward contract has the same shape as the payoff of the underlying asset. This means that for every dollar increase/decrease in the value of the underlying at maturity, the payoff of the forward will increase/decrease with the same dollar amount.
- Non-linear products are derivatives whose payoff function is a non-linear function i.e. the payoff of the contract is a non linear function of the sport price of the underlying at maturity.

1.12 Future contracts

- A futures contract is an exchange traded, standardized forward contract (standardized amount, quality, delivery location, etc.) There are some important differences with forward contracts:
- Forward contracts are OTC products; futures are exchange traded This means that forwards can be tailor-made; futures are standardized.
- Forwards can not be unwound unilaterally, futures can: one can easily take the opposite position and close out the original position while doing so.
- Forwards put default risk on the individual parties, futures assign the default risk (credit risk) to the exchange: the original contract between the long and the short is split into two separate contracts; which are both made with the exchange. Every party that enters a future contract, has the exchange as its counterparty. In this way, credit risk is reduced.
- Credit risk for forward contracts is managed with collateral agreements. Future contracts use margin accounts. Every profit and every loss that a party makes is transferred to the counterparty, within the same day. In this way credit risk is avoided for both parties. Every day, there is a settlement.

1.13 Swaps

- In a swap-agreement there are two counterparties i.e. swaps are bilateral contracts. The two counterparties exchange a series of cashflows. The the cashflow series that are described in the contract are called the 'leg of the swap.' Swaps have a linear payoff profile which means that swaps are linear products. Standardized swap agreements do not require upfront payment to enter into the swap position.

1.14 Options

- There are two possible positions in an option contract:
  - A long position. This is the party that buys the option contract.
  - A short position. This is the party that sells an option contract.
- An option is a financial derivative product which entails a right for one party and an obligation for the other.
  - From the point of view of the long position i.e. the buyer of the option, an option entails a right.
  - From the point of view of the short position i.e. the seller of the option, an option entails an obligation.
- There exist different kinds of option contracts. They differ in the obligation/right they grant to the party that is short/long:
  - Options that give the long party the right to buy the underlying are called call options. This means the short has the obligation to sell the underlying when the option is exercised.
  - Options that give the long party the right to sell the underlying are called put options. This means the short has the obligation to buy the underlying when the option is exercised.

1.15 Options and forward contracts compared

- A forward contract entails the obligation to buy or sell the underlying at maturity, at a prefixed price. This obligation is completely unconditional. Whatever happens; the long will have to buy the underlying and the short will have to sell the underlying.
- Going short in an option contract also gives rise to an obligation. However this obligation is conditional. When the long party in the option contract does not want to exercise his right to buy or sell the underlying; he is not obligated to do so. This means that going short in an option contract entails taking on a conditional obligation. If it is not in the advantage of the long to exercise the option, the short will not have to do anything at all.

- We have to pay attention to the terminology.
  - In the case of option contracts, going long refers to buying the option contract. The long party has the right to buy or sell the underlying. Going short refers to selling the option contract. The short party has the conditional obligation to buy or sell the underlying.
  - In the case of forward contracts, going long refers to entering a position where one would benefit from a rise in the price of the underlying. Going short refers to entering a position where one would benefit from a decline in the price of the underlying.

1.16 Plain vanilla options

- A plain vanilla option is an option that gives the buyer of this option the right to:
  - Buy (call option) or to sell (put option),
  - a certain amount of underlying (the contract size),
  - at an agreed price (the strike price or exercise price),
  - at maturity date (European style) or up to maturity date (American style).
- Plain vanilla options are also called standard European style and American style calls and puts. Exotic options are options that differ from plain vanilla options in one or more ways.

1.17 Time of exercise

- Option contract can differ in terms of when they can be exercised by the long party. We discern the following broad categories:
  - European style options i.e. options that give the right to exercise, only at maturity
  - American style options i.e. options that can be exercised at any moment during the maturity period.
  - Bermudian style options i.e. options that can be exercised at specific moment or during specific period.

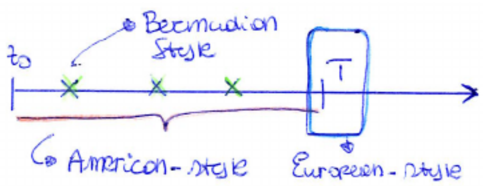


Figure 1.7: Different option styles.

1.18 Types of Options

- There exist different types of options. Some examples are listed below.
  - Options as such.
  - Embedded options i.e. options embedded in another contract or product. Examples are callable bonds and convertible bonds mortgages.
  - Real options.

1.19 Applications of option contracts

- Using options is 'pretty powerful' because they allow the user to enter positions with high leverage.
- Consider the example of a European call option on an underlying where the spot price of the underlying at the time t_0 is equal to \$100. The spot price of the underlying at the time t_1 is equal to \$120. The option premium is equal to \$5. The long pays the option premium while the short receives the option premium.
  - The gross return on the underlying when going long in the spot market is given by:
  $$\frac{\$120}{\$100} - 1 = 20\%$$
  - The return on a European call option, when going long in the option market is given by:
  $$\frac{\$120 - \$100 - \$5}{\$5} = \frac{\$15}{\$5} = 300\%$$
  - It is clear that options are financial instruments that enable taking positions with high leverage. We cannot forget however that leverage is a two-edged sword. for example, if the value of the underlying S_{t_1} becomes \$80, the gross return on the long call option becomes:
  $$\frac{\$0 - \$5}{\$5} = -100\%$$
  - The gross return on a long spot position in the underlying under the same circumstances is given by:
  $$\frac{\$80}{\$100} - 1 = -20\%$$
  - It is clear leverage works both ways.

1.20 The moneyness of an option

- The term moneyness refers to how the strike price of an option is related to the spot price of the underlying asset.
- The degree of moneyness M is defined as the ratio of the spot price of the underlying S_t over the strike price of the option K. Mathematically:

$$M = \frac{S_t}{K}$$

- An option is:
  1. In the money.
     - If immediate exercise would lead to a positive cashflow.
     - For a call option, this means: $K < S_t$.
     - For a put option, this means: $K > S_t$.
  2. Out of the money.
     - If immediate exercise would lead to a negative cashflow.
     - For a call option, this means: $K > S_t$.
     - For a put option, this means: $K < S_t$.
  3. At the money.
     - if immediate exercise would lead to a zero cashflow.
     - This means: $K = S_t$.

1.21 Option premia

- An option premium is the current market price of an option contract. It is thus the income received by the seller (writer) of an option contract to another party. An option premium is quoted per unit of the underlying.

1.22 Payoff from an option contract

- First we discuss the payoff of the long call option. The payoff of a contract is defined as the net cashflow generated by the option contract, at maturity date or also as the value of the contract at maturity date. The call option gives the long position the right to buy the underlying value at the fixed exercise price K, at maturity T.
  - Consider the example of a long call option where the exercise price K is equal to \$100
    1. First, consider the case where the spot price at maturity S_T is greater than the exercise price K. Suppose for example that the spot price at maturity is equal to \$120. Mathematically:

       $$S_T > K$$
       $$\$120 > \$100$$

       The long could buy the underlying at \$100 and immediately sell on the market at \$120, thus realizing a payoff of \$20. Mathematically:

       $$\begin{aligned} S_T - K &= \$120 - \$20 \\ &= \$20 \end{aligned}$$

       In this case, the payoff consists of the difference between the spot price at maturity and the exercise price.

       $$f_T^{Long\ call} = S_T - K$$

    2. Next, consider the case where the spot price at maturity S_T is lower than the exercise price K. Suppose for example that the spot price at maturity is equal to \$75. Mathematically:

       $$S_T < K$$
       $$\$75 < \$100$$

       The option won't be exercised: the option owner let's the option expire. The payoff is zero in this case:

       $$f_T^{Long\ call} = 0$$

  - The payoff of the long call option is therefore given by:

$$f_T^{Long\ call} = \begin{cases} 0, & K > S_T \\ S_T - K, & K < S_T \end{cases}$$

– We can now construct the payoff diagram for the long call option of our example. Graphically, this becomes:

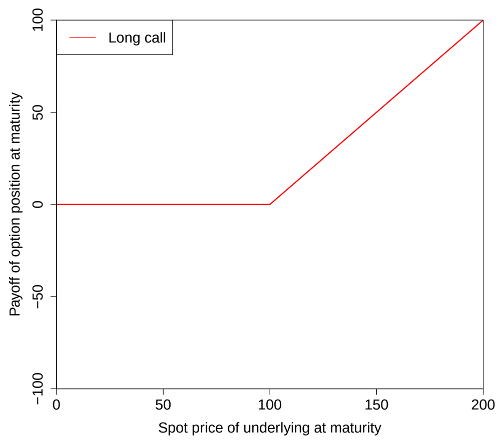


Figure 1.8: Payoff diagram for the long call option.

– A long call option is a bullish instrument: one speculates on the upside. If the underlying goes up in value, the speculator wins. If the underlying goes down in value, the speculator loses. There is no limit on the upside: as long as the value of underlying goes up in value, the payoff will rise.

– We see the linear relationship in the payoff diagram from the forward contract: in a payoff diagram, a long forward contract and a long call option have the same linear relationship if $S_T > K$.

– However, the option contract dominates the forward contract. With a long forward contract, the delivery price is set in such a way that on t_0 the contract has no value for both parties. Hence, there is no cashflow on t_0. The long forward can win or lose. The long call option cannot lose. When $S_T < K$ he lets the option expire. Because the value of the forward is zero at t_0, the value of the option needs to be positive at t_0. One will need to pay a premium to acquire a long call option. This premium is called to the short party in the option contract.

- The differences between the long call option and the long forward contract become clear when comparing their payoff diagrams:

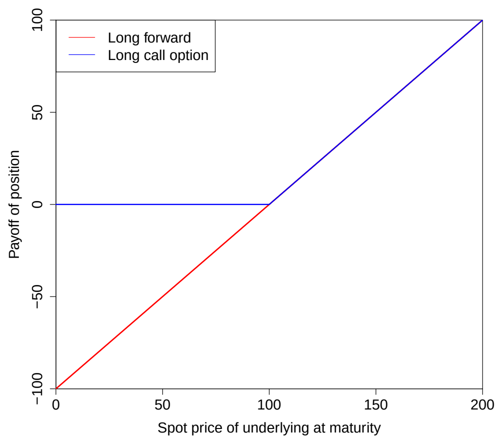


Figure 1.9: Payoff profiles of the long call option and the long forward contract.

- We now discuss the payoff from a short call option.
  - Options are a zero sum game; the payoff diagram of a short call option therefore is the mirror image of the payoff diagram for a long call option on the same underlying.

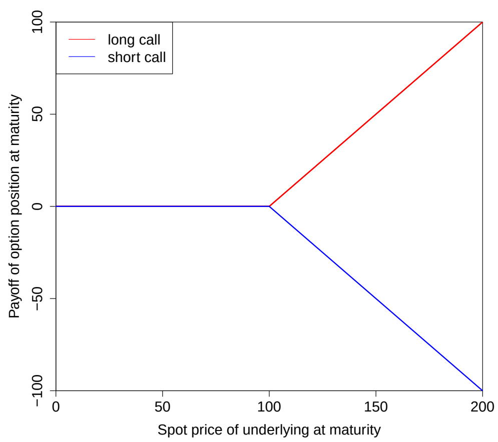


Figure 1.10: Payoff diagrams of the long and the short call option.

- Next, consider the payoff from a long put option. A long put option gives the owner the right to sell the underlying at a certain, fixed price.
  - Consider the example of a long put option where the exercise price K is equal to \$100.
    1. First, consider the case where the spot price at maturity S_T is lower than the exercise price K. Suppose for example that the spot price of the underlying at maturity is equal to \$80. Mathematically:

       $$\begin{aligned} S_T &< K \\ \$80 &< \$100 \end{aligned}$$

       The long could buy the underlying at \$80 on the spot market and immediately exercise his option and sell the underlying at \$100, thus realizing a payoff of \$20. In this case, the payoff consists of the difference between the exercise price and the spot price. Mathematically:

       $$\begin{aligned} f_T^{\textit{Long put}} &= \$100 - \$80 \\ &= \$20 \end{aligned}$$

    2. Next, consider the case where the spot price at maturity S_T is greater than the exercise price K. Suppose for example that the spot price of the underlying at maturity is equal to \$120. Mathematically:

       $$\begin{aligned} S_T &> K \\ \$80 &> \$100 \end{aligned}$$

       The option won't be exercised: the option owner let's the option expire. The payoff is zero in this case:

       $$f_T^{\textit{Long put}} = 0$$

  - The payoff of the long put option is therefore given by

    $$f_T^{\textit{Long put}} = \begin{cases} 0, & S_T > K \\ K - S_T, & S_T < K \end{cases}$$

– We can now construct the payoff diagram for the long put option. From this we can see the linear relationship in the payoff diagram from the short forward contract. In a payoff diagram, a short forward contract and a long put option have the same linear relationship if $S_T < K$ Graphically, this becomes:

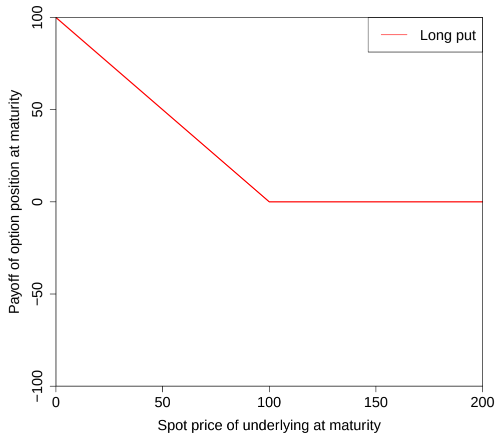


Figure 1.11: Payoff diagram of a long put option.

–

- Finally, consider the short put option.
  – Again the payoff diagram for the short party is the mirror image of the payoff diagram for the long party.

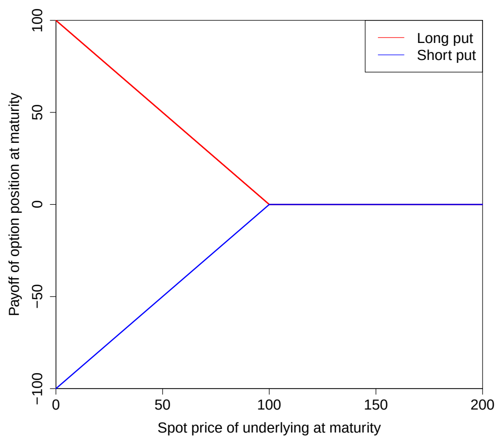


Figure 1.12: The payoff diagrams for the short put option and the long put option.

- We now summarize our finding graphically and mathematically.

$$f_t^{long\,call} = max\ (S_T - K, 0)$$
$$f_t^{long\,put} = max\ (K - S_T, 0)$$

Where:

K = The strike price.
T = The maturity date.
S = The spot price.
$f_T^{Position}$ = The value of contract at maturity T.

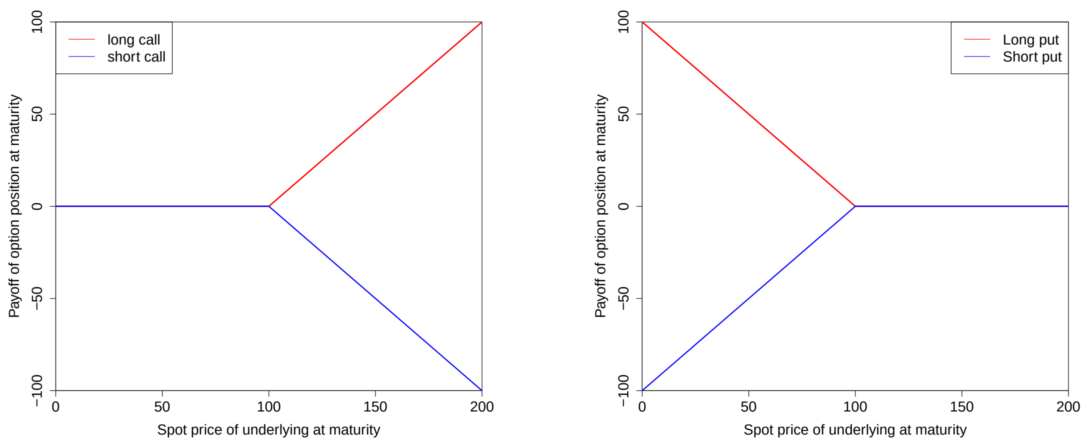


Figure 1.13: Payoff diagrams for options contracts. Left: the long call and the short call option. Right: the long put and the short put options.

1.23 Profit diagrams

- When acquiring an option, you have to pay an option premium. To acquire a profit diagram you must subtract the option premium from the value of the option at a certain point in time.
- A profit diagram for a derivative is nothing else than a payoff diagram that undergo a vertical shift reflecting the premium that is paid and other potential costs or benefits. Mathematically, this becomes:

$$Profit_T^{Long} = f_T^{Long} - premium$$
$$Profit_T^{Short} = f_T^{Short} + premium$$

- The option holder (long) will break even when the long regains his paid premium i.e. the short loses his received premium.
- However, we neglected the time value of money. The payoff is realized at maturity date and the premium is paid when purchasing the option. In order to correct this, you have to compound or discount one of the cash-values.

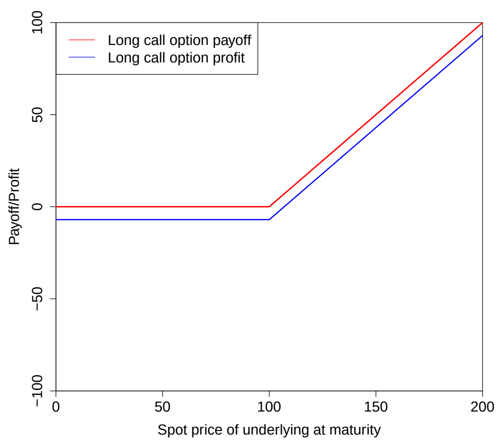


Figure 1.14: Profit and payoff diagram of the long call option compared.

1.24 Applications of derivative products

- Derivatives can be used in different applications. We will discuss arbitrage, speculation and hedging with derivative products.
- The first application we consider is hedging. Consider the example where a US company will receive 100 million Euro in March. We investigate what options this firm has to hedge their exposure to the Dollar/Euro exchange rate.
  - The firm could opt to leave its exposure open by doing nothing. The firm then has an exposure and bears a risk. Leaving a position open, leaves open a risk. The firm gains if the USD decreases in value and loses if the USD increases in value.
  - The firm could also opt to use future contracts to hedge its exposure. It could do this by entering a short position in Euro futures. The underlying of such a Euro future contract is a set amount in Euro. The future price for the underlying is expressed in USD. By entering the short forward contract, we fix a price in USD for the underlying. We now consider the different possible outcomes.
    1. First, suppose the EUR increases in value vis à vis the USD. This means that the spot price at maturity of one Euro is higher than the spot price at the inception of the forward contract. In that case, the payoff of the option contract will be negative.

$$S_{t_0}^{Euro} < S_T^{Euro}$$
$$f_T^{Short\ fut.} < 0$$

    2. Next, suppose the EUR decreases in value vis Ã vis the USD. This means that the spot price at maturity of one Euro is lower than the spot price at the inception of the forward contract. In that case, the payoff of the option contract will be positive.

$$S_{t_0}^{Euro} > S_T^{Euro}$$
$$f_T^{Short\ fut.} > 0$$

  - The firm could also opt to use option contracts to hedge its exposure. The firm would want to buy put options i.e. it would take a long position in put options. The underlying is a set amount in Euro. The price for the underlying is expressed in USD. The firm therefore has an option to sell the underlying. We fix a price in USD for the underlying. We now consider the different possible outcomes.

1. First, suppose the EUR increases in value vis à vis the USD. In that case, the firm will not exercise the option but rather sell the underlying spot.

$$S_{t_0}^{Euro} < S_T^{Euro}$$
$$f_T^{EUR\, long\, put} = 0$$

2. Next, suppose the EUR decreases in value vis Ã vis the USD. In this case, the firm will exercise the option and realize a positive payoff.

$$S_{t_0}^{Euro} > S_T^{Euro}$$
$$f_T^{EUR\, long\, put} > 0$$

– Observe that one could also:

1. Enter a long position in dollar futures.
2. Enter a long position in call dollar options.

- The next application is speculation. Speculation is all about creating an exposure to a certain risk, with regards to a certain event. Consider the example where a US company believes the JPY will appreciate vis-à-vis the USD over the next three months. We investigate what options this firm has to open an exposure to the JPY.
  – The firm could choose to buy the JPY in the spot market. In this case there is no leverage.
  – The firm could also opt to take a long position in futures on the JPY . In this case there is leverage.
  – Finally, the firm could choose to enter a position of long call options on the JPY.

1.25 Structured products

- Structured products are portfolios of existing financial products (including derivatives). The goal is to create new payoff profiles by combining these different financial products. Examples of structured products are:
  - Packages of forwards and options, combined into one product.
  - Bonds with embedded options.
- Consider the following example where we try to deconstruct a structured product, based on the payoff profile of this structured product.
  - The payoff profile is given by the following set of equations:

$$f_T^{StrProd} = \begin{cases} F - S_t & \text{if } F > S_T \\ m \cdot (F - S_T) & \text{otherwise} \end{cases}$$

    Where:

$$\begin{aligned} F = & \text{ the contracted forward price.} \\ S_T = & \text{ the spot price at maturity.} \\ m = & \text{ a multiplier, indicating leverage.} \end{aligned}$$

  - If $S_T < F$, the payoff profile is a 45° line which corresponds to the payoff profile as an ordinary short forward. If $S_T > F$, the slope of the payoff profile is steeper as the slope of an ordinary short forward; for every dollar the underlying increases, the payoff decreases with more than one dollar i.e. $m > 1$. Mathematically:

$$f_T^{StrProd} = \begin{cases} F - S_T & when\ \ S_T < F \\ F - m \cdot S_T & when\ \ S_T > F \end{cases}$$

  - We now investigate how we could construct such a product. This is discussed in more detail in courses such as financial engineering. We now try to construct a target forward with m=2.
    1. We use a short forward; the payoff profile is given by:

$$f_T^{Short\,forward} = F - S_T$$

    2. We use a short call; the payoff profile is given by:

$$f_T^{Short\,call} = \begin{cases} 0 & when\ \ F > S_T \\ F - S_T & when\ \ F < S_T \end{cases}$$

3. We combine these two products:

$$f_T^{Target\,forward} = f_T^{Short\,call} + f_T^{Short\,forward} = \begin{cases} F - S_T & when\;\; F > S_T \\ 2 \cdot (F - S_T) & when\;\; F < S_T \end{cases}$$

   Now there is a \$2 decrease of the payoff for every increase of the spot price at maturity S_T with \$1.

- We now consider an example for what is called a range forward or flexible forward.
  - Suppose the payoff profile of this range forward is given by the set of equations below. The payoff follows S_T, whenever you are between two boundaries. When you exit those two boundaries, you reach a built-in cap, floor. Deconstructing this payoff profile is a nice exercise.

$$f_T^{Range\;fw.} = \begin{cases} \$1.57/GBP & when\;\; S_T < 1.57 \\ S_T & when\;\; 1.57 < S_T < 1.64 \\ \$1,64 & when\;\; 1.64 < S_T \end{cases}$$

2 Fixed Income Analytics

In this chapter, we take a look at some fixed income financial instruments. We start by introducing the different types of interest rates. We then take a look at the term structure of interest rates. This chapter corresponds with chapter 5 of the book.

Fixed income analytics is all about the time value of money:

- We calculate the future value of a cashflow sum by compounding that cashflow.
- We calculate the present value of a future cashflow by discounting that cashflow.

There are some conventions that are used when discounting or compounding. For instance:

- Day count basis conventions.
- Conventions on how interest on interest is treated i.e. capitalization.

2.1 Day count conventions

Interest rates on a loan or investment will typically be expressed as a fraction of a duration of time. This duration of time is called the interest base period N.

Typically, interest rates are expressed as a fraction of a year. In that case, n equals the number of interest-bearing days and where N equals the number of days in a year. The term τ of the loan or investment can then be expressed in the following manner:

$$\tau = \frac{n}{N}$$

- The numerator n is equal to the number of days between the start date t_0 and the maturity date T. It is also called the number of interest-bearing days.

$$n = T - t_0$$

- The denominator N equals the number of days in a year.

There are multiple ways in which the value of the numerator n and the denominator N can be determined. These are so-called day-count conventions. There are 6 commonly used day-count conventions:

- $\frac{ACT}{ACT}$
- $\frac{ACT}{365}$
- $\frac{ACT}{360}$
- $\frac{30}{360}$
- $\frac{30E}{360}$
- $\frac{30E+}{360}$

ACT denotes actual. In this case, we use the actual number of days when computing the term loan. When ACT is used in the numerator, we count the exact number of days between the start date and the maturity date. This interest type convention is denoted with the term **exact time**. When ACT is used in the denominator, we use the average number of days of each year to which the loan applies. This interest type convention is denoted with the term **exact interest**.

When 30 is used in the numerator, we speak of **approximate time**. In this case, if the day of the month of the start date is equal to 31, we change it to 30. Furthermore, the day of the month of the maturity date is changed:

- 30/360: if day of month of maturity date is equal to 31 and day of month of start date is equal to 30 or 31, then we assume day of month of maturity date is equal to 30.
- 30E/360: if day of month of maturity date is equal to 31, then we assume day of month of maturity date is equal to 30.
- 30E+/360: if day of month of maturity date is equal to 31, then we assume day of month of maturity date is equal to 1 of the next month.

When 365 is used in the denominator, we assume that every year consists of 365 days. In contrast, when 360 is used in the denominator, we assume that every year consists of 360 days. In this case, we speak of **ordinary interest**.

2.2 Simple interest

Interest is added over time but only to the starting capital i.e. interest is calculated on the starting capital. The interest I is proportional to the term τ. This means the interest I changes at the same rate as the term τ. The interest I over a capital sum C, outstanding during a period τ is given by:

$$I = C_{t_0} \cdot (i \cdot \tau)$$

It is clear that the interest is proportional to the term τ.

2.2.1 Future value of a capital sum

At the end of a period τ, a capital sum C, outstanding at a simple interest rate i, will have generated an interest equal to:

$$I = C_{t_0} \cdot i\tau$$

This interest is added to the initial capital C_{t_0}. The sum of both is called the accumulated amount of capital or the future value of the capital sum: Mathematically:

$$\begin{aligned} C_T &= C_{t_0} + I \\ &= C_{t_0} + C_{t_0} \cdot i\tau \end{aligned}$$

In summary, we know that:

$$\begin{gathered} C_T = C_{t_0} + I \\ I = C_{t_0} \cdot (i\tau) \end{gathered}$$

From these expressions, it must follow that:

$$C_T = C_{t_0} \cdot (1 + i\tau)$$

2.2.2 Bankers discount

- The present value of a capital sum or a cash flow is calculated by discounting that cash flow.
- A discount or banker's discount E is defined as a reduction that is allowed when one pays of their debt before the end of the term, before maturity. It is a reduction of the redeemable sum.

The discount rate d is defined as a percentage that is applied to the terminal value of a loan/investment. Mathematically:

$$E = C_T \cdot d\tau$$

Suppose:

$$\begin{aligned}
&C_T = \text{the terminal value of the debt.}\\
&E = \text{the discount.}\\
&C_{t_0} = \text{the discounted value of the debt.}\\
&\tau = \text{the term of the debt.}
\end{aligned}$$

It must be true that:

$$E = C_T - C_{t_0}$$

The discount is thus the difference between the nominal value C_T and the discounted value C_{t_0} of the debt.

2.2.3 Present value of a capital sum

To calculate the present value of a cash flow, one needs to discount that cash flow. The discounted value of a capital sum C_{t_0} is given by:

$$\begin{aligned}
C_{t_0} &= C_T - E\\
&= C_T - C_T \cdot d\tau\\
&= C_T \cdot (1 - d\tau)
\end{aligned}$$

2.2.4 Equivalent interest rate

We are interested in the interest rate that corresponds with a given discount rate. To derive this interest rate, we equate the interest that is generated over the whole term I to the discount over the whole term. Mathematically, this becomes:

$$\begin{aligned} I &= E \\ C_{t_0} \cdot i \cdot \tau &= C_T \cdot d \cdot \tau \end{aligned}$$

We also know that:

$$C_T = \frac{C_{t_0}}{1 - d\tau}$$

An expression for computing the present value of a cash flow can ve easily derived from the expression for computing the future value of a cash flow.

$$\begin{aligned} C_{t_0} &= \frac{C_T}{1 + r\tau} \\ C_T &= C_{t_0} \cdot (1 + r\tau) \end{aligned}$$

Combining the statements above we acquire the following equation for the interest rate that corresponds with a given discount rate:

$$\begin{aligned} C_{t_0} \cdot i \cdot \tau &= \frac{C_{t_0}}{1 - d\tau} \cdot d\tau \\ i &= \frac{d}{1 - d\tau} \end{aligned}$$

2.3 Intermediate payments

In some cases, debtors have the right to make early repayments. If so, the outstanding debt and the interest due have to be recalculated. This recalculation can be done using two approaches.

- Merchant rules. All cash flows (positive and negative) are compounded up to maturity T. These cash flows are then summed at maturity T. This sum reflects the remaining outstanding capital amount that will need to be repaid.
- Declining balance rules. In this case, every cash flow is split up in an amortization amount and an interest amount. We can then recalculate the outstanding capital amount.

2.4 Compound interest

The case where interest is paid periodically and added to the capital sum is denoted with the term compound interest. The underlying assumption is that the accrued interest is reinvested at the same interest rate until maturity. Id est, interest payments accrue interest themselves until maturity.

2.4.1 Future value of a capital sum

Suppose interest capitalization takes place every next period (for instance, every year). I.e. interest is added to the capital sum at the end of each period.

During the first period, interest is earned only on the capital sum. We can therefore calculate the future value of the capital sum after one period using the formulas for simple interest. The accumulated capital after one period C_{t_1} is equal to:

$$C_{t_1} = C_{t_0} \cdot (1 + i)$$

During the next period, interest is received not only on the capital sum but also on the interest that was earned in the first period. Id est the interest is calculated on the accumulated capital after period one. The accumulated capital after two periods is therefore equal to:

$$C_{t_2} = C_{t_1} \cdot (1 + i)$$

We substitute the equation for C_{t_1} into the equation for C_{t_2}:

$$\begin{aligned} C_{t_2} &= C_{t_0} \cdot (1 + i) \cdot (1 + i) \\ &= C_{t_0} \cdot (1 + i)^2 \end{aligned}$$

We can easily generalize the above statement for n periods instead of two periods:

$$C_{t_n} = C_{t_0} \cdot (1+i)^n$$

The factor $(1+i)$ is sometimes called the accumulation factor. We may think of the accumulation factor as a growth factor because $(1+i) > 1$. The future value of the capital sum C_T grows exponentially with the number of periods n. Id est C_T is an increasing power function of the number of periods n.

2.4.2 Present value of a capital sum

We can also easily derive a formula for the present value of a capital sum C_{t_0}. We already know the formula for the future value of a capital sum:

$$C_{t_n} = C_{t_0} \cdot (1+i)^n$$

We can therefore easily derive an expression for C_{t_0}. It is clear that the present value of the capital sum C_{t_O} decreases exponentially with the number of years.

$$\begin{aligned} C_{t_0} &= \frac{C_{t_n}}{(1+i)^n} \\ &= C_{t_n} \cdot (1+i)^n \end{aligned}$$

2.4.3 The discount rate and the equivalent interest rate

We are also interested in the interest rate that corresponds with a given discount rate. As is the case for simple interest, the discount rate d is applied to the terminal value of a loan or investment.

$$C_{t_o} = C_{t_n} \cdot (1-d)^n$$

It must follow that:

$$\begin{aligned} \frac{C_{t_n}}{C_{t_0}} &= \frac{C_{t_0} \cdot (1+i)^n}{C_{t_0}} \\ &= (1+i)^n \end{aligned}$$

We can easily deduce that, in the case of compound interest, the discount factor is defined as $(1-d)^n$:

$$\begin{aligned} \frac{C_{t_0}}{C_{t_n}} &= \frac{C_{t_n} \cdot (1-d)^n}{C_{t_n}} \\ &= \frac{1}{(1+d)^n} \end{aligned}$$

So the following must also hold:

$$
\begin{aligned}
1+i &= \frac{1}{1-d} - \frac{1-d}{1-d} \\
&= \frac{d}{1-d}
\end{aligned}
$$

Or equivalently:

$$
\begin{aligned}
1+i &= \frac{1}{1-d} \\
1-d &= \frac{1}{1+i} \\
d &= \frac{-1}{1+i} + 1 \\
&= \frac{-1+1+i}{1+i} \\
&= \frac{i}{1+i}
\end{aligned}
$$

The equation above gives us the interest rate that corresponds with a given discount rate, in the case of periodic compounding. Note that the resulting expression is the same as for the case case of simple interest.

2.4.4 Total amount of interest earned

We can also easily calculate the total amount of interest that is earned during n years I_n by subtracting the starting capital C_{t_0} from the accumulated amount of capital after n years C_{t_n}. Mathematically:

$$\begin{aligned} I_n &= C_{t_n} - C_{t_0} \\ &= C_{t_0} \cdot (1+i)^n - K_{t_0} \\ &= C_{t_0} \cdot \big[(1+i)^n - 1\big] \end{aligned}$$

2.4.5 Nominal and effective interest rates

Interest compoundings will typically be expressed as a fraction of a duration of time. This duration of time is called the compounding base period M.

In the case of effective interest rates, the compounding base period M is equal to the interest base period N. That is, the compounding frequency or the number of interest compoundings per interest base period is equal to one. Mathematically:

$$\begin{cases} N = \text{interest base period} \\ M = \text{compounding base period} \\ \dfrac{N}{M} = m = \text{compounding frequency} \end{cases}$$

For example, the effective annual interest rate, annual equivalent rate (AER) is the annual interest rate on a loan or financial product where compound interest is payable annually in arrears.

Suppose interest is being capitalized at the end of every period equal to $\frac{1}{m}^{th}$ of the interest base period N. The corresponding nominal interest rate is denoted by $i(m)$.

In order to use our previously derived formulas for compounding and discounting, we will need to align the interest base period N with the compounding base period M. Mathematically:

$$i = i(m)/m$$

To calculate the future value the future value of a capital sum C_{t_n}; we will need to take into account the total number of capitalizations at the effective interest rate i. We can then calculate the total number of capitalization periods as $m \cdot n$.

In conclusion, there are $(m \cdot n)$ capitalizations, each at interest rate $\frac{i(m)}{m}$.

After n years, the future value of a capital sum C_{t_n} will equal:

$$C_{t_n} = C_{t_0} \cdot \left[1 + \frac{i(m)}{m}\right]^{m \cdot n}$$

With:

$$\begin{aligned} m &= \text{the number of capitalizations per base period.} \\ n &= \text{the number of base periods.} \\ i(m) &= \text{the nominal interest rate with compounding frequency } m. \end{aligned}$$

An expression for the effective interest rate can be derived by equating the future values of the capital sum C_{t_0} for the nominal and the effective interest rate:

$$C_{t_0} \cdot (1 + i)^n = C_{t_0} \cdot \left[1 + \frac{i(m)}{m}\right]^{m \cdot n}$$

From this, it follows that:

$$i = (1 + \frac{i(m)}{m})^m - 1$$

Conversely, we can deduce a nominal interest rate $i(m)$, from an effective interest rate i:

$$i(m) = m \cdot \left[(1 + i)^{\frac{1}{m}} - 1\right]$$

Notes

2.5 Continuous interest

Suppose we let the number of interest capitalizations increase to infinity: $m \to \infty$. This means that the interest is added to the outstanding capital in a continuous matter. The interest trickles into the capital sum continuously. The interest rate per period $\frac{i(m)}{m}$ will go to zero. This is because every time period is infinitesimally small and the amount added to the capital is negligibly small. The nominal continuous interest $i(m = \infty) = \lim_{m\to\infty} i(m)$ will however be finite. We call $i_\infty = \delta$ the force of interest or the nominal rate of interest per unit time, momentarily convertible.

2.5.1 Future value of a capital sum

We derive the formula for the future value of a capital sum C_T by solving the limit case for the future value for compound interest.

$$\begin{aligned} C_T &= \lim_{m\to\infty} \left[C_{t_0} \cdot (1 + \frac{i(m)}{m})^{m\tau} \right] \\ ln(C_T) &= \lim_{m\to\infty} \left[ln\{C_{t_0} \cdot (1 + \frac{i(m)}{m})^{m\tau}\} \right] \end{aligned}$$

First, consider taking the logarithm of both sides of the equation.

$$\begin{aligned} ln\{C_{t_0}(1 + \frac{i(m)}{m})^{m\tau}\} &= ln\{C_{t_0}\} + ln\{(1 + \frac{i(m)}{m})^{m\tau}\} \\ &= ln\{C_{t_0}\} + m\tau \cdot ln\{1 + \frac{i(m)}{m}\} \\ &\Big| \; ln\{1 + h\} \approx h \; for \; small \; values \; of \; h \\ &= ln\{C_{t_0}\} + m\tau \cdot \frac{i(m)}{m} \\ &= ln\{C_{t_0}\} + \tau \cdot i(m) \end{aligned}$$

Then, take the limit of both sides of the equation for $m \to \infty$.

$$\begin{aligned} ln\{C_T\} &= \lim_{m\to\infty} \left[ln\{C_{t_0}\} + \tau \cdot i(m) \right] \\ &= ln(C_{t_0}) + \tau\delta \end{aligned}$$

$$C_T = C_{t_0} \cdot e^{\delta\tau}$$

2.5.2 Present value of a capital sum

Using the formula for the future value of a capital sum, we can easily deduce the present value of a capital sum in the case of continuous interest:

$$C_T = C_{t_0} \cdot e^{\delta\tau}$$

Becomes:

$$C_{t_0} = C_T \cdot e^{-\tau\delta}$$

2.5.3 The discount rate and the equivalent interest rate

Notes

2.6 Annuities

2.6.1 Definition

An annuity is a series of future payments, for which the time interval Δt between those points (also called "the period") is constant. In general a period covers one year, hence the name "annuity."

2.6.2 The multiplicative nature of money

We deposit a sum of money with present value PV in a bank that pays interest at the rate r. After one year the deposited sum of money has grown to $PV(1+r)$. We call this amount the future value FV. We may write this future value as:

$$FV = PV \cdot (1+r)$$

We May also think of (1+r) as a growth factor. Continuing this process for another year, compounding the interest annually, the future value FV will become:

$$FV = [PV \cdot (1+r)] \cdot (1+r) = PV \cdot (1+r)^2$$

If we continue this process, compounding the sum for a total of n years, the future value becomes:

$$FV = PV \cdot (1+r)^n$$

Principle of consistency

The principle of consistency states that in a consistent market, the proceeds of an investment strategy where the investor invests an amount A over a time span between t_0 and t_2 should be equivalent to an investment strategy where the investor invests an amount A over a time span between t_0 and t_1 and an amount A over a time span between t_1 and t_2. The course of action taken by the investor should not influence the proceeds of the investment.

$$A(t_0, t_2) = A(t_0, t_1) \cdot A(t_1, t_2)$$

2.6.3 Value additivity

The value of an asset V_0 equals the present value of all future cashflows CF_k that the owner of the asset expects to receive during the lifetime of the asset:

$$V_0 = \frac{CF_1}{(1+r)^1} + \frac{CF_2}{(1+r)^2} + ... + \frac{CF_T}{(1+r)^T}$$

With:

$$\begin{aligned} V_0 &= \text{the value of the asset at time 0} \\ CF_k &= \text{the cashflow in year t} \\ r &= \text{the appropriate discount rate} \end{aligned}$$

2.6.4 Types of annuities

We already stated that annuity is a series op payments where the time between the payments is constant. However there exist different types of annuities. These annuities differ with regard to the cashflow amounts, the maturity and the timing of the cash flows. We now list different types of annuities:

- Level annuity: the annuity amount is constant:

$$CF_1 = CF_2 = ... = CF_n$$

- Varying annuity: payments are not all of the same size:

$$CF_1 \neq CF_2 \neq ... = CF_n$$

- Temporary annuity or annuity certain: the number of payments is finite:

$$n \in \mathbb{R}$$

- Perpetuity: the number of payments is infinite:

$$n = \infty$$

- Deferred annuity: the annuity is delayed with m periods: the first payment takes place in the period τ_m.
- Annuity due, annuity payable in advance: the payments are made at the beginning of each period.
- Ordinary annuity or annuity payable in arrears: the payments are made at end of each period.

2.6.5 General case annuity

We consider the most general case of an annuity where there are n payments:

$$A_{t_0}, A_{t_1}, A_{t_2}, ..., A_{t_j}, ..., A_{t_n}$$

Present value

The present value is given by:

$$\begin{aligned} V_0 &= A_{t_0} + A_{t_1} \cdot \frac{1}{(1+i)^1} + ... + A_{t_j} \cdot \frac{1}{(1+i)^j} + ... + A_{t_n} \cdot \frac{1}{(1+i)^n} \\ &= \sum_{k=1}^{n} \left(A_{t_k} \cdot \frac{1}{(1+i)^k} \right) \end{aligned}$$

Future value

The future value is given by:

$$\begin{aligned} V_n &= A_{t_0} \cdot (1+i)^n + A_{t_1} \cdot (1+i)^{n-1} + ... + A_{t_j} \cdot (1+i)^{n-j} + ... + A_{t_n} \cdot (1+i)^0 \\ &= \sum_{k=0}^{n} (A_{t_k} \cdot (1+i)^{n-k}) \\ &= (1+i)^n \cdot \sum_{k=0}^{n} (A_{t_k} \cdot (1+i)^{-k}) \end{aligned}$$

Or equivalently:

$$\begin{aligned} &= (1+i)^n \cdot \sum_{k=0}^{n} (A_{t_k} \cdot (\frac{1}{1+i})^k) \\ &= (1+i)^n \cdot V_0 \end{aligned}$$

2.6.6 Level annuities certain

A level annuity certain had the following characteristics:

- Payments are all of the same amount.
- The number of payments is finite.
- The annuity starts immediately.

We make use of the following symbols in our derivations:

- a: the present value of a level ordinary annuity of \$1
- s: the future value of a level ordinary annuity of \$1
- $\ddot{a}$: the present value of a level annuity due of \$1
- $\ddot{s}$: the future value of a level annuity due of \$1

We also make use of the following notation for discount factors:

$$v^j = \frac{1}{(1+i)^j}$$

And of the following notation for the compound factors:

$$u^j = (1+i)^j$$

The term of the annuity is indicated by a subscript on the right, placed under a right angle. For example, the present value of an n-year; immediate; level; ordinary annuity, of \$1 is indicated by:

$$a_{\overline{n}|}$$

The deferral period is indicated by a subscript on the left, followed by a vertical bar. For example, the present value of an n-year; m-year-deferred; level; ordinary annuity, of 1\$ is indicated by:

$$_{m|}a_{\overline{n}|}$$

Present value

We will now calculate the present value and the future value of a level ordinary annuity

$$A = \begin{cases} A_0 = 0 \\ A_1 = A_2 = ... = A_n = \$1 \end{cases}$$

Suppose v is the discount factor and u is the capitalization factor so that:

$$v^j = \frac{1}{(1+i)^j}$$

$$u^j = (1+i)^j$$

For the present value, we find that:

$$\begin{aligned} a_{\overline{n}|} &= \sum_{j=1}^{n} v^j \cdot 1 \\ &= v + v^1 + v^2 + ... + v^n \\ &= \frac{(1-v)}{(1-v)} \cdot v + v^1 + v^2 + ... + v^n \\ &= \frac{(v^1 + v^2 + ... + v^n - v^2 - ... - v^{n+1})}{(1-v)} \\ &= \frac{(v - v^{n+1})}{(1-v)} \\ &= \frac{v(1-v^n)}{v(\frac{1}{v} - 1)} \\ &= \frac{1-v^n}{u-1} \\ &= \frac{1-v^n}{i} \end{aligned}$$

Future value

For the future value, we find that:

$$
\begin{aligned}
s_{\overline{n}|} &= \sum_{j=1}^{n} 1 \cdot u^{n-j} \\
&= u^n \cdot \sum_{j=1}^{n} u^{-j} \\
&= u^n \cdot \sum_{j=1}^{n} v^{j} \\
&= u^n \cdot a_{\overline{n}|} \\
&= u^n \cdot (\frac{1 - v^n}{i}) \\
&= \frac{u^n - 1}{i}
\end{aligned}
$$

2.6.7 Deferred annuity

Consider a level, n-year, m-year deferred ordinary annuity certain, of \$1 The current value of the annuity at time $t = m$ is equal to:

$$a_{\overline{n}|}$$

Therefore, the current value at time t_0, the present value of the deferred annuity ${}_{m|}a_n$ is given by:

$$\begin{aligned} {}_{m|}a_{\overline{n}|} &= v^m \cdot a_{\overline{n}|} \\ & v^m \cdot \frac{1 - v^m}{i} \\ &= \frac{v^m - v^{m+n}}{i} \end{aligned}$$

For a deferred perpetuity, the present value becomes:

$${}_{m|}a_{\overline{\infty}|} = \frac{v^m}{i}$$

The future value of a deferred annuity is equal to that of the corresponding immediate annuity:

$${}_{m|}s_{\overline{n}|} = s_{\overline{n}|}$$

2.6.8 Annuity due

A annuity due is equivalent to an ordinary annuity whose starting point has been advanced by one period. That is, a one year advanced ordinary annuity.

The present value is given by:

$$\begin{aligned} \ddot{a}_{\overline{n}|} &= {}_{-1|}a_{\overline{n}|} \\ &= v^{-1} \cdot a_{\overline{n}|} \\ &= u \cdot a_{\overline{n}|} \end{aligned}$$

In more general terms:

$$\begin{aligned} {}_{m|}\ddot{a}_{\overline{n}|} &= u \cdot {}_{m|}a_{\overline{n}|} \\ &= v^{m-1} \cdot a_{\overline{n}|} \end{aligned}$$

For the future value of an annuity due, the following statements are true:

$$\ddot{s}_{\overline{n}|} = u \cdot s_{\overline{n}|}$$

and

$${}_{m|}\ddot{s}_{\overline{n}|} = u \cdot s_{\overline{n}|}$$

2.6.9 Continuously payable annuities

2.6.10 Varying annuities

2.6.11 Uncertain annuities

2.7 Bonds

We now take a look at bonds. A bond is a fixed income instrument that represents a loan made by an investor to a borrower (typically corporate or governmental). A bond could be thought of as an I.O.U. between the lender and borrower that includes the details of the loan and its payments. Bonds are used by companies, municipalities, states, and sovereign governments to finance projects and operations. Owners of bonds are debtholders, or creditors, of the issuer.

2.7.1 The theoretical price of a bond

We want to determine the price of a bond. The theoretical price of a bond corresponds to the sum of the discounted cashflows of the bond. First, we consider zero coupon bonds or for short zero bonds. A zero-coupon bond is a debt security that does not pay interest but instead trades at a deep discount, rendering a profit at maturity, when the bond is redeemed for its full face value. To determine the theoretical price of a zero bond we have to discount the nominal value of the bond over the time to maturity T. Mathematically:

In the case of simple interest, the price of the bond is given by:

$$V_0 = \frac{CF_T}{1+r} \cdot \frac{d}{B} = \frac{CF_T}{1+r} \cdot \tau$$

In the case of compound interest, the price of the bond is given by:

$$V_0 = \frac{CF_T}{(1+r)^\tau}$$

In the case of periodically compounded interest, the price of the bond is given by:

$$V_0 = \frac{CF_T}{(1+\frac{r}{m})^\tau}$$

In the case of continuous interest, the price of the bond is given by:

$$V_0 = CF_T \cdot e^{-r\tau}$$

Or, in general:

$$V_0 = CF_T \cdot u$$

With:

- t_0: the starting date.
- T: the maturity date.
- τ: the time to maturity.
- CF_T: the cashflow at maturity.

- m: the number of discount periods in a year.
- r: the appropriate discount rate.
- $\frac{d}{B}$: the number of days divided by the number of days in a year.
- u: a discount factor.

2.7.2 Market quotations

Bonds can be quoted in several ways. A quotation can be either a yield or a price. Given a yield we can always determine a corresponding price.

Quotation on a discount basis

Bonds can be quoted using a discount basis. The yield quotation on a discount basis is defined as the face value of the bond minus the price of the bond as a percentage of the face value of the bond, discounted over the maturity of the bond using a certain discount basis. The yield is mathematically defined as:

$$\gamma_d = \frac{F-P}{F} \cdot \frac{B}{n}$$

With

- F: the face value of the bond.
- P: the price of the bond.
- B: the discount basis.
- n: the number of calendar days to maturity.
- $(F-P)$: the net cashflow at maturity.
- $\frac{B}{n}$: a discount factor.
- γ_d: the yield calculated using a certain discount basis B.

It follows that the price of the bond is given by:

$$P = F \cdot (1 - \frac{1 \cdot \gamma_d}{B})$$

Money market basis

Bonds can also be quoted on a money market basis. The yield on a money market basis γ_m, is defined as the face value of the bond minus the price of the bond, as a percentage of the price of the bond, discounted over the maturity of the bond using a certain discount basis. The yield is mathematically defined as:

$$\gamma_m = \frac{F-P}{P} \cdot \frac{B}{n}$$

It follows that the price of the bond is given by:

$$P = \frac{F}{1 + \frac{n \cdot \gamma_m}{B}}$$

The zero coupon yield to maturity

To define the Yield To Maturity we need to know about the Internal Rate of Return or IRR for short. The IRR is the discount rate for which the Net Present Value or NPV for short of the bond is equal to zero. The Net Present Value or the NPV for short is the sum of all discounted future cashflows of a bond. NPV-calculations rely on a discount rate that is available a priori. In a mathematical expression, the IRR is defined as:

$$\sum_t \frac{CF_t}{(IRR)^t} = 0$$

The yield to maturity or YTM for short is the total return anticipated on a bond if the bond is held until it matures. The yield to maturity is expressed as an annual rate. In other words, it is the internal rate of return (IRR) of an investment in a bond if the investor holds the bond until maturity, with all payments made as scheduled and reinvested at the same rate.

Example: An investor receives \$100 in one year for an investment of \$97.50. Calculate the yield per annum with annual, sem-annual and continuous compounding.

- Annual compounding

$$\begin{aligned} FV &= PV \cdot (1 + \gamma_{ac})^{\tau} \\ \gamma_{ac} &= (\frac{FV}{PV})^{\frac{1}{\tau}} - 1 \\ &= (\frac{100}{97.50})^{\frac{1}{0.25}} - 1 \\ &= 0.10657674 \end{aligned}$$

- Semi-annual compounding

$$\begin{aligned} FV &= PV \cdot (1 + \frac{\gamma_{sac}}{m})^{\tau m} \\ \gamma_{sac} &= m * (\frac{FV}{PV})^{\frac{1}{\tau * m}} - 1 \\ &= 2 * (\frac{100}{97.50})^{\frac{1}{0.25 * 2}} - 1 \\ &= 0.103879027 \end{aligned}$$

- Continuous compounding

$$
\begin{aligned}
FV &= PV \cdot e^{\gamma_{cc}\tau} \\
\gamma_{cc} &= \log(\frac{FV}{PV}) * \frac{1}{\tau} \\
&= \log(\frac{100}{97.50}) * \frac{1}{0.25} \\
&= 0.101271232
\end{aligned}
$$

The spot rate

The spot rate or the zero rate of an investment with maturity T, is the interest rate one earns on an equivalent investment that provides a payoff only at maturity T. We denote the spot rate until time T as:

$$s_T$$

2.7.3 The term structure of interest rates

The term structure of interest rates is the relationship between interest rates or bond yields and different terms or maturities. Id est a term structure of interest rates, depicts the interest rates of similar bonds at different maturities. When graphed, the term structure of interest rates is known as a yield curve. The term structure thus gives the interest rate an investor expects on an investment that only provides a single cash stream on maturity, for different maturities.

When calculating the theoretical price of a bond we need to take into account the term structure of interest rates. Often, a single interest rate is used when discounting the cashflows of a bond or some other financial instrument. However, each cashflow should in theory be discounted using the zero rate for the maturity of that cashflow. This is what we will be doing in the future.

Maturity (in years)	Zero rate (c.c.)
0.5	5.0
1	5.8
1.5	6.4
2	6.8

Figure 2.1: An example of a term structure of interest rates

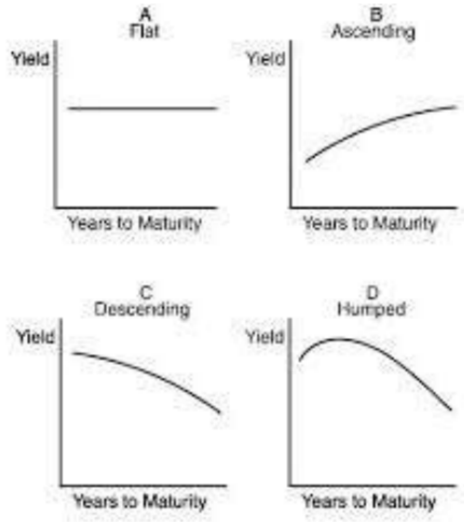


Figure 2.2: A term structure of interest rates can boast different shapes

- Example:
  - Given the previous term structure, we can calculate the theoretical price of a two year bond with a face value of \$100 providing a coupon of 6% semi-annually. We do this by discounting every cash stream against the corresponding zero rate:

$$V_0 = 3 * e^{-0.05*0.5} + 3 * e^{-0.058*1.0} + 3 * e^{-0.06*1.5} + 103 * e^{-0.068*2.0}$$
$$= 98.39$$

2.7.4 Spot rates and forward rates

- A spot rate S_{t_1} is an interest rate for an investment with maturity t_1 that starts today at t_0.
- A forward rate $f_{t_{0,t_1 \to t_2}}$ is an interest rate for an investment with maturity T that starts at some point in time in the future t_1. A forward rate is thus an interest rate that can be contracted now, but that is applicable to a future time period.
- Both spot rates and forward rates and forward rates concern investments with one single payoff at maturity. Id est spot rates and forward rates are yields on zero coupon bonds.

Computing the forward rate

Forward rates can be computed from spot interest rates through a process called bootstrapping. We consider the following example:

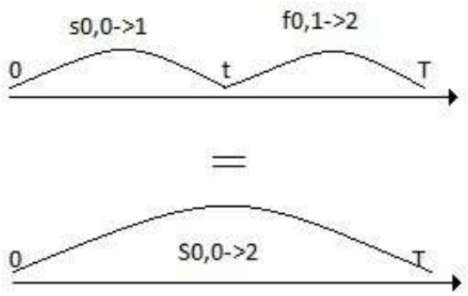


Figure 2.3: Future and forward rates

In the first figure we fix an interest rate at t_0 for the first period and simultaneously fix an interest rate for the second period. In the second example we fix an interest rate at t_0 for the two periods combined. The interest that is paid in both situations must be equal. Mathematically, this becomes:

$$Interest(Period\ 1) + Interest(Period\ 2) = Interest(Period\ 1 + Period\ 2)$$

Expresses mathematically, this becomes:

$$A \cdot e^{(1-0)\cdot s_{t_0,t_0 \to t_1}} \cdot e^{(2-0)\cdot f_{t_0,t_1 \to t_2}} = A \cdot e^{(2-0)\cdot s_{t_0,t_0 \to t_2}}$$

We can rearrange the equation:

$$f_{t_0,t_1 \to t_2} = \frac{s_{t_0,t_0 \to t_2} \cdot (2-0) - s_{t_0,t_0 \to t_1} \cdot (1-0)}{2-1}$$

We can generalize and simplify this expression:

$$f_{0,t \to T} = \frac{T \cdot s_T - t \cdot s_t}{T - t}$$

With:

- T: the maturity that matches with the forward rate.
- t: the start date that matches with the forward rate; or the maturity that matches with the spot rate.
- S_T: the spot rate that matches with maturity T.
- s_t: the spot rate that matches with maturity t.

We now look at the same example, but we now use annual compounding; we can easily see that:

$$A \cdot (1 + s_{t_0, t_0 \to t_1},)^{(1-0)} \cdot (1 + f_{t_0, t_1 \to t_2})^{(2-1)} = A \cdot (1 + s_{t_0, t_0 \to t_2})^{(2-0)}$$

We can rearrange the equation:

$$f_{t_0, t_1 \to t_2} = \sqrt[2-1]{\frac{(1 + s_{t_0, t_0 \to t_2})^{(2-0)}}{(1 + s_{t_0, t_0 \to t_1})^{(1-0)}}} - 1$$

We can generalize this expression:

$$f_{t_0, t \to T} = \sqrt[T-t]{\frac{(1 + s_T)^T}{(1 + s_t)^t}}$$

With:

- T: the maturity that matches with the forward rate.
- t: the start date that matches with the forward rate; or the maturity that matches with the spot rate.
- S_T: the spot rate that matches with maturity T.
- s_t: the spot rate that matches with maturity t.

Computing the spot rate

Naturally, we can also compute the spot rate for a certain maturity T, starting from the 'immediate spot rate' and the subsequent forward rates.

We will again start with the case for continuous interest. We already saw that:

$$A \cdot e^{(1-0)\cdot s_{t_0,t_0\to t_1}} \cdot e^{(2-0)\cdot f_{t_0,t_1\to t_2}} = A \cdot e^{(2-0)\cdot s_{t_0,t_0\to t_2}}$$

From this we can easily find the following expression for the spot rate $s_{t_0,t_0\to t_2}$:

$$s_{t_0,t_0\to t_2} = \frac{s_{t_0,t_0\to t_1} + f_{t_0,t_1\to t_2}}{2}$$

We can also see that for the case of three consecutive periods the following statement holds:

$$A \cdot e^{(1-0)\cdot s_{t_0,t_0\to t_1}} \cdot e^{(2-0)\cdot f_{t_0,t_1\to t_2}} \cdot e^{(3-2)\cdot f_{t_0,t_2\to t_3}} = A \cdot e^{(3-0)\cdot s_{t_0,t_0\to t_3}}$$

Again, we find the expression for the spot rate $s_{0,0\to 3}$:

$$s_{t_0,t_0\to t_3} = \frac{s_{t_0,t_0\to t_1} + f_{t_0,t_1\to t_2} + f_{t_0,t_2\to t_3}}{3}$$

Again, we can generalize and simplify the expression above. In general, the spot rate is the average of the short term spot rate and the consecutive forward rates:

$$s_n = \frac{1}{n} \cdot (s_1 + f_{t_0,t_1\to t_2} + f_{t_0,t_2\to t_3} + ... + f_{t_0,t_{n-1}\to t_n})$$

Exercise

Maturity in years	*Spot rate*	*Forward rate*
1	10%	
2	10.5%	
3	10.8%	
4		11.6%

Figure 2.4: Compute the missing forward and spot rates.

Maturity in years	*Spot rate*	*Forward rate*
1	10%	
2	10.5%	11%
3	10.8%	11.4%
4	11%	11.6%

Figure 2.5: Solution.

We rewrite the equation that expresses the relation between the forward and the spot rate. Doing so will allow us to solve the exercise.

$$
\begin{aligned}
f_{0,t\to T} &= \frac{T \cdot s_T - t \cdot s_t}{T - t} \\
&= \frac{T \cdot s_T - t \cdot s_t - t \cdot s_T + t \cdot s_T}{T - t} \\
&= s_T \cdot (\frac{T - t}{T - t}) + (s_T - s_t) \cdot (\frac{t}{T - t}) \\
&= s_T + (s_T - s_t) \cdot (\frac{t}{T - t})
\end{aligned}
$$

Key takeaways

- A spot rate is a contracted price for a transaction that will start immediately.
- A forward rate is a contracted price for a transaction that will start at an agreed upon date in the future.
- The spot rate typically is used as the starting point for negotiating the forward rate.

Forward discount factors

We will now derive a formula for the forward discount factor. In general, a discount factor is the inverse of the capitalization factor:

$$
df_{T_{0,t\to T}} = e^{-1\cdot(T-t)\cdot f_{0,t\to T}}
$$

We also know that:

$$
f_{t_{0,t\to T}} = \frac{T \cdot s_T - t \cdot s_t}{T - t}
$$

We substitute this expression into the first equation

$$
\begin{aligned}
df_{t_{0,t\to T}} &= e^{-1\cdot(T-t)\cdot\frac{T\cdot s_T - t\cdot s_t}{T-t}} \\
&= e^{s_t\cdot t - s_T\cdot T} \\
&= \frac{e^{-s_T\cdot T}}{e^{-s_t\cdot T}} \\
&= \frac{df_T}{df_t}
\end{aligned}
$$

The forward discount factor that applies to the period $[t, T]$ is the discount factor for the period $[0, T]$ divided by the discount factor for the period $[0, t]$.

We now want to check if the expression above is also valid in case of periodic compounding. We already know that:

$$\begin{cases} \frac{df_T}{df_t} & = \frac{\frac{1}{(1+s_T)^T}}{\frac{1}{(1+s_t)^t}} \\ df_{t_0,t\to T} & = \frac{1}{(1+f_{0,t\to T})^{(T-t)}} \end{cases}$$

If the proposed expression is valid, it must hold that:

$$\begin{aligned} \frac{1}{(1+f_{t_0,t\to T})^{(T-t)}} &= \frac{df_T}{df_t} = \frac{\frac{1}{(1+s_T)^T}}{\frac{1}{(1+s_t)^t}} \\ &= \frac{(1+s_t)^t}{(1+s_T)^T} \\ (1+f_{0,t\to T})^{(T-t)} &= \frac{(1+s_T)^T}{(1+s_t)^t} \\ f_{0,t\to T} &= \sqrt[T-t]{\frac{(1+s_T)^T}{(1+s_t)^t}} - 1 \end{aligned}$$

This is the expression we already found for the future rate in the case of periodic compounding, thus we showed the expression from which we started is also valid in the case of periodic compounding.

Computing the forward rate from the forward discount factor

We now want to compute the forward rate, starting from the forward discount factor. We start with the case of continuous compounding. We already knew that:

$$df_{t_0,t\to T} = e^{-(T-t)\cdot f_{0,t\to T}}$$

By taking the logarithm of both sides of the equation, we arrive at:

$$ln(df_{t_0,t\to T}) = -(T-t)\cdot f_{t_0,t\to T}$$

We rearrange the expression:

$$f_{t_0,t\to T} = \frac{-ln(df_{t_0,t\to T})}{T-t}$$

Or:

$$f_{t_0,t\to T} = \frac{-1}{T-t}\cdot ln(\frac{df_T}{df_t})$$

The instantaneous forward rate

We will now compute the instantaneous forward rate. This is a limit case of the forward rate. We defined the forward rate as:

$$f_{t0,t \to T} = s_T + (s_T - s_t) \cdot (\frac{t}{T-t})$$

We take the limit, where T approaches t, of both sides of the equation:

$$\lim_{T \to t} \left(f_{t0,t \to T} \right) = \lim_{T \to t} \left(s_T + (s_T - s_t) \cdot (\frac{t}{T-t}) \right)$$

We get the following result:

$$\lim_{T \to t} \left(f_{t0,t \to T} \right) = s_T + \frac{ds}{dt} \cdot t$$

Sensitivity of the forward rate

We are interested in the change of the forward rate with regards to the spot rate.

Change in Spot Rate	1.00%

Time	Spot Rate	Discount Factor	Forward Rate
2	5%	0.9048	
2.5	5.20%	0.8781	6.00%
2	5%	0.9048	
2.5	5.25%	0.8770	6.26%
Proc. Change	1.00%	-0.13%	4.33%

Figure 2.6: Sensitivity of the term curve.

Suppose, that at time 2 the spot rate is 5%; and that at time 2.5 the spot rate is 5.2%; we can easily calculate the discount factors:

$$df_2 = e^{-0.05\cdot 2} = 0.904837418$$
$$df_{2.5} = e^{-0.052\cdot 2.5} = 0.878095431$$

Suppose the spot rate is 1% higher at time 2 and 1% higher at time 2.5:

$$s_2 = 0.05 \cdot (1.01) = 5.05\%$$
$$s_{2.5} = 0.052 \cdot (1.01) = 5.252\%$$

We can recalculate the discount factors for the higher spot rates:

$$df_2 = e^{-0.0505\cdot 2} = 0.903933033$$
$$df_{2.5} = e^{-0.0552\cdot 2.5} = 0.871098692$$

We can calculate the forward rate for the case where $s_2 = 5\%$ and $s_{2.5} = 5.2\%$:

$$f_{0,t\to T} = \frac{-1}{T-t} \cdot ln(\frac{df_T}{df_t})$$
$$f_{0,2\to 2.5} = \frac{-1}{2.5-2} \cdot ln(\frac{0.8781}{0.9048})$$
$$f_{0,2\to 2.5} = 0.06$$

We can also calculate the forward rate for the case where $s_2 = 5.05\%$ and $s_{2.5} = 5.252\%$:

$$f_{0,t\to T} = \frac{-1}{T-t} \cdot ln(\frac{df_T}{df_t})$$
$$f_{0,2\to 2.5} = \frac{-1}{2.5-2} \cdot ln(\frac{0.8770}{0.9048})$$
$$f_{0,2\to 2.5} = 0.0624$$

A percentual change of 1% in the spot rate at time 2 and at time 2.5 leads to a change in the forward rate of:

$$\% - change = \frac{0.0624}{0.06} - 1 = 4\%$$

2.7.5 Coupon bonds

We will now take a look at coupon bonds. Coupon bonds are bonds that provide periodic cashflows or coupons to the investor.

Pricing a coupon bond

A coupon bond offers cashflows at different pints in time. To price a coupon bond we need to discount and sum these cashflows. To discount a cashflow one needs an interest rate. Remember that we have to use different interest rate when discounting cashflows with different maturities. As stated before, the relationship between the zero rate of an investment and the time to maturity of that investment is given by the term structure of interest rates.

As stated before, we determine the theoretical price of a coupon bond by discounting the different cashflows the coupon bond provides throughout its lifetime. For the case of continuous compounding, the price of a coupon bond is given by:

$$P_0 = \sum_{t=1}^{T} \left(CF_t \cdot e^{-s_t \cdot t}\right)$$

For the case of periodic compounding, this price is given by:

$$P_0 = \sum_{t=1}^{T} \left(\frac{CF_T}{(1 + \frac{s_t}{m})^{t \cdot m}}\right)$$

We just multiply every cashflow with a discount factor which is the inverse of the corresponding capitalization factor and sum all these cashflows.

Example

We want to determine the price of a coupon bond, starting from the following term structure.

Maturity in years	*Spot rate (ACT/ACT, cc, in %)*
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

Figure 2.7: Term structure of a bond

This term structure gives the spot rate of similar zero bonds with different maturities. We can use the term structure to calculate the present value of cashflows with different maturities.

A coupon bond is analogous with a portfolio of:

- Several zero bonds that each have a cashflow equal to the coupon.
- A zero bond that has a cashflow of the nominal value of the coupon bond.

The theoretical price of a two-year bond, providing a 6% coupon semi-annually is equal to:

$$P_0 = 3 \cdot e^{-0.05 \cdot 0.5} + 3 \cdot e^{-0.058 \cdot 1} + 3 \cdot e^{-0.064 \cdot 1.5} + 3 \cdot e^{-0.068 * 2} = \$98.39$$

Pricing a risky coupon bond

We now consider a risky coupon bond id est a bond where the interest rate is subject to risk. This means the future cashflows are not certain. We want to determine the price of such a bond. The theoretical price of a risky coupon bond is given by:

$$P_0^{AA} = \sum_{t=1}^{T} \frac{CF_T}{(1 + s_t + CS_t^{AA})^t}$$

With:

- CS_t^{AA}: the credit spread at moment t
- P_0^{AA}: the price of the risky coupon bond at t_0

Computing spot rates for coupon bonds

To price a coupon bond we need the term structure of interest rates. This means we need the spot rates for different maturities. Using a method called bootstrapping we can construct a zero-coupon-bond term structure.

When considering a zero bond, determining the zero rate is easy. All we need to do is express the equation for the price of a zero bond in terms of the interest rate s_T we use to discount the nominal value of the bond:

$$P_{t_0} = P_T \cdot e^{-s_T * T}$$
$$s_T = \frac{1}{T} \cdot ln(\frac{P_T}{P_{t_0}})$$

However,if we want to determine the zero rate of a coupon bond, we have to take the coupons with the different maturities into account. We will have to discount these coupons using the zero rate that corresponds to the maturity of these coupons.

One can obtain the zero rates for the maturities of the different coupons by looking at:

- Zero bonds with the same maturity
- Coupon bonds with the same maturity, for which we already know the zero rates for the maturities of the different coupons.

Constructing a term structure of interest rates is therefore a gradual process where we use other bonds that have a maturity that corresponds to the maturity of one of the coupons.

Example

We want to price a coupon bond. First, we construct a term structure of interest rates for the maturities that correspond with the maturities of the different coupons of the bond. We do this by looking at other bonds with maturities that correspond to the coupons of the cashflows of our bond.

The example is as follows: we consider the following sample of coupon bonds. A coupon is paid every six months (Ãf the bond includes a coupon)

Bond Principal (dollars)	Time to Maturity (years)	Annual Coupon (dollars)	Bond Price (dollars)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

Note: The stated coupon is assumed to be paid every six months.

Figure 2.8: The sample of coupon bonds.

In the first place, we determine the zero rate for a three month maturity. In this case, determining the zero rate is easy because we are given a zero-coupon bond with a maturity of three months. The zero rate for a maturity of three months can thus be derived in the following manner:

$$
\begin{aligned}
P_T &= P_{t_0} \cdot e^{s_T * \tau} \\
100 &= 97.5 \cdot e^{s_T * \tau} \\
100 &= 97.5 \cdot e^{s_T \cdot 0.25} \\
ln(\frac{100}{97.5}) &= s_T \cdot 0.25 \\
4 \cdot 0.025 &= s_T \\
s_{(0.25)} &= 0.1013
\end{aligned}
$$

In the same way, we can determine the zero rate for a maturity of six months:

$$
\begin{aligned}
100 &= 94.9 \cdot e^{s_T * \tau} \\
100 &= 94.9 \cdot e^{s_T \cdot 0.50} \\
s_{(0.5)} &= 0.1047
\end{aligned}
$$

The zero rate for a maturity of one year is given by:

$$
\begin{aligned}
100 &= 90 \cdot e^{s_T \cdot T} \\
100 &= 90 \cdot e^{s_T \cdot 1.0} \\
s_{(1)} &= 0.1054
\end{aligned}
$$

The zero rate for a maturity of one and a half year is somewhat more difficult to determine. We are given a coupon bond with a maturity of one and a half year. To determine the zero rate we will need to discount the different coupons using the corresponding zero rate. The coupons are granted every six months. Luckily we already determined the zero rates for maturities of six and twelve months above. The zero rate for a maturity of eighteen months can be determined in the following manner:

$$
\begin{aligned}
P_0 &= \sum_{t=1}^{T} \left(CF_t \cdot e^{-s_t \cdot t} \right) \\
96 &= 4 \cdot e^{-s_{(0.5)} \cdot 0.5} + 4 \cdot e^{-s_{(1)} \cdot 1} + (100 + 4) \cdot e^{-s_{(1.5)} \cdot 1.5} \\
96 &= 4 \cdot e^{-0.10469 \cdot 0.5} + 4 \cdot e^{-0.10536 \cdot 1} + (100 + 4) \cdot e^{-s_{(1.5)} \cdot 1.5} \\
96 &= 3.769 + 3.6000 + 104 \cdot e^{-s_{(1.5)} \cdot 1.5} \\
96 - 7.3960 &= 104 \cdot e^{-s_{(1.5)} \cdot 1.5} \\
ln(0.85196) &= -s_{(1.5)} \cdot 1.5 \\
\frac{-0.1602}{1.5} &= -s_{(1.5)} \\
s_{(1.5)} &= 0.1068
\end{aligned}
$$

We used the zero rates for bonds with maturities of six months and one year to discount the coupons of this bond. This allowed us to also determine the zero rate for this bond. This process is called bootstrapping.

The zero rate for a maturity of two years is determined in the same way:

$$
\begin{aligned}
P_0 &= \sum_{t=1}^{T} \left(CF_t \cdot e^{-s_t \cdot t} \right) \\
101.6 &= 6 * e^{-s_{(0.5)} \cdot 0.5} + 6 \cdot e^{-s_{(1)} \cdot 1} + 6 \cdot e^{-s_{(1.5)} \cdot 1.5} + 106 \cdot e^{-s_{(2)} \cdot 2} \\
101.6 &= 6 \cdot e^{-0.10469 \cdot 0.5} + 6 \cdot e^{-0.10536 * 1} + 6 \cdot e^{-0.1068_{(1.5)} \cdot 1.5} + 106 \cdot e^{-s_{(2)} \cdot 2} \\
ln(\frac{85.3941}{106}) &= -s_{(2)} \cdot 2 \\
s_2 &= 0.1080
\end{aligned}
$$

Using these spot rates, we can plot a term structure also known as a yield curve:

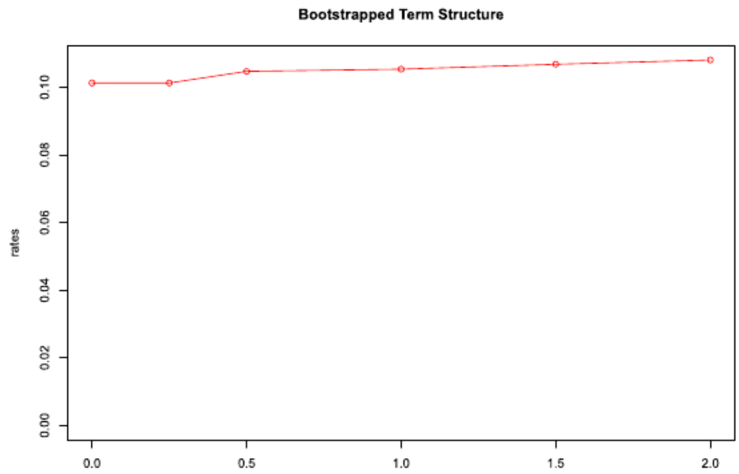


Figure 2.9: The bootstrapped term structure.

Splines

A spline is a function that knits piecewise functions together in an way that ensures continuity of the function, of its slope and of the slope of the slope.

- The piecewise functions used can have various forms (e.g. quadratics, cubics, exponentials,...)
- We need to chose the number of knots and their location arbitrarily
- More knots implies greater accuracy but also a greater computational burden
- Rule of tumb: $\sqrt{n}$ (McCullouch) or $\frac{1}{3}$ (Fisher)

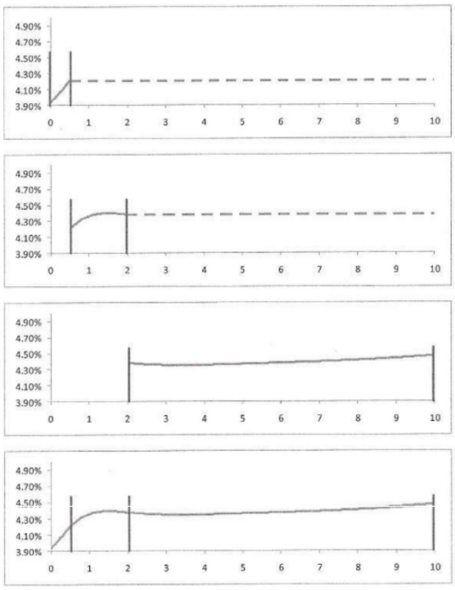


Figure 2.10: Three piecewise functions and a corresponding spline

We will now try to span the term structure (the yield curve) with five cubic equations with some yet unknown parameters.

The spot rate of a zero bond with maturity t is given by the following five equations:

$$s_t = \begin{cases} a_1 + b_1 t + c_1 t^2 + d_1 t^3 & \text{when} \quad 0 < t \leq 0.5 \\ a_2 + b_2(t-0.5) + c_2(t-0.5)^2 + d_2(t-0.5)^3 & \text{when} \quad 0.5 < t \leq 1 \\ a_3 + b_1(t-1) + c_3(t-1)^2 + d_3(t-1)^3 & \text{when} \quad 1 < t \leq 2 \\ a_4 + b_1(t-2) + c_4(t-2)^2 + d_4(t-2)^3 & \text{when} \quad 2 < t \leq 5 \\ a_5 + b_1(t-5) + c_5(t-5)^2 + d_5(t-5)^3 & \text{when} \quad 5 < t \leq 10 \end{cases}$$

We impose that the different equations for s_t are equal at the knots i.e. the zero rates have to be equal for maturities where two piecewise functions meet. Mathematically, we impose the following constraints:

$$
\begin{aligned}
&a_1 + b_1 t + c_1 t^2 + d_1 t^3 = a_2 + b_2(t-0.5) + c_2(t-0.5)^2 + d_2(t-0.5)^3 && \text{at} \quad t = 0.5 \\
&a_2 + b_2(t-0.5) + c_2(t-0.5)^2 + d_2(t-0.5)^3 = a_3 + b_1(t-1) + c_3(t-1)^2 + d_3(t-1)^3 && \text{at} \quad t = 1 \\
&a_3 + b_1(t-1) + c_3(t-1)^2 + d_3(t-1)^3 = a_4 + b_1(t-2) + c_4(t-2)^2 + d_4(t-2)^3 && \text{at} \quad t = 2 \\
&a_4 + b_1(t-2) + c_4(t-2)^2 + d_4(t-2)^3 = a_5 + b_1(t-5) + c_5(t-5)^2 + d_5(t-5)^3 && \text{at} \quad t = 5
\end{aligned}
$$

Evaluating these constraints at the can derive the following constraint for the different parameters:

$$
\begin{aligned}
a_2 &= a_1 + \frac{1}{2}b_1 + \frac{1}{4}c_1 + \frac{1}{8}d_1 \\
a_3 &= a_2 + \frac{1}{2}b_2 + \frac{1}{4}c_2 + \frac{1}{8}d_2 \\
a_4 &= a_3 + b_3 + c_3 + d_3 \\
a_5 &= a_4 + 3b_4 + 9c_4 + 27d_4
\end{aligned}
$$

Taking the first derivative and evaluating at the intersections gives us the following constraints for the parameters, after rearranging the equations: gives us the following constraints:

$$
\begin{aligned}
b_2 &= b_1 + c_1 + \frac{3}{4}d_1 \\
b_3 &= b_2 + c_2 + \frac{3}{4}d_2 \\
b_4 &= b_3 + 2c_3 + 3d_3
\end{aligned}
$$

Taking the second order derivative of the constraint, evaluating at each intersection and rearranging the equation gives us the following constraints:

$$
\begin{aligned}
c_2 &= c_{1+\frac{3}{2}}d_1 \\
c_3 &= c_2 + \frac{3}{2}d_2 \\
c_4 &= c_3 + 3d_3 \\
c_5 &= c_4 + 9d_4 \\
b_5 &= b_4 + 6c_4 + 27d_4
\end{aligned}
$$

We can then calculate the value of each parameter, taking into account the different constraints. The result is shown by the figure on the next page.

t	0.08	0.17	0.25	0.33	0.42	0.50	0.58	0.67	0.75	0.83	0.92	1	1.25	1.5	2	3	4	5	8	9	10
Spot rates	4.982	4.955	4.926	4.883	4.840	4.797	4.760	4.724	4.694	4.673	4.654	4.637	4.432	4.442	4.444	4.444	4.444	4.438	4.412	4.364	4.307
Fitted	4.983	4.930	4.891	4.864	4.845	4.832	4.768	4.738	4.704	4.667	4.626	4.578	4.555	4.513	4.395	4.414	4.402	4.457	4.416	4.369	4.302

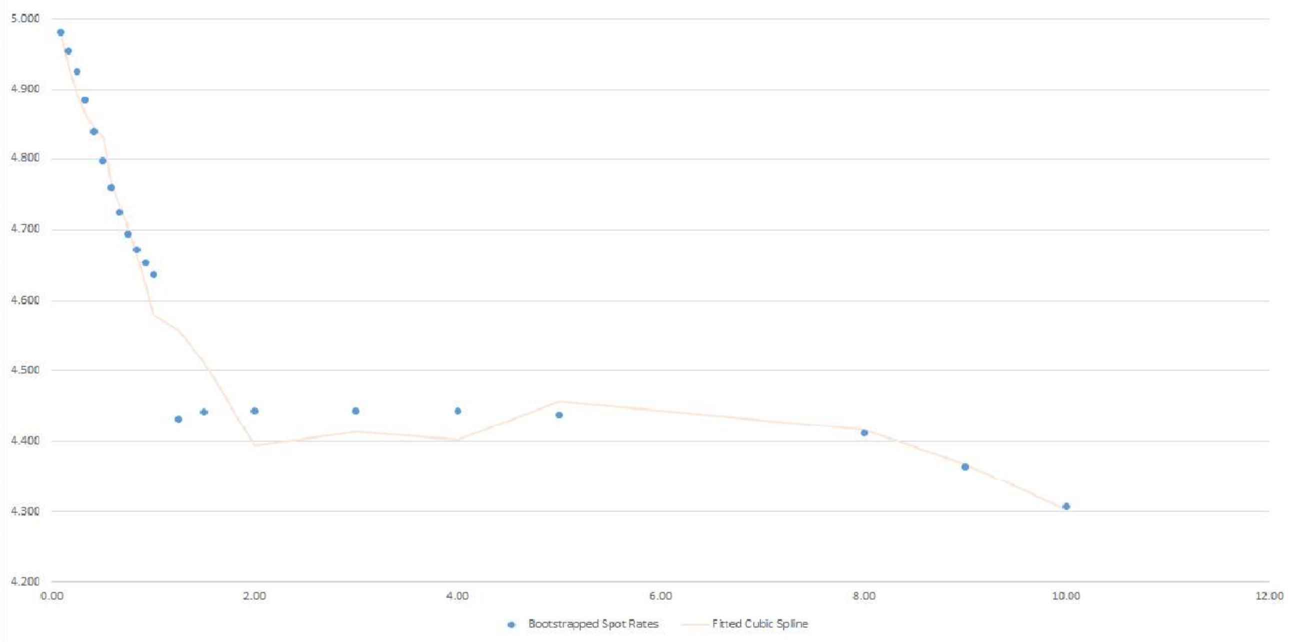


Figure 2.11: The fitted cubic spline versus the bootstrapped spot rates.

Bond yields

Bonds can be quoted using different yield. In this section we will define and discuss different possible yields for bonds.

- The current yield
  - The current yield is defined as:

$$Current\ Rate = \frac{Annual\ Coupon}{Current\ Price}$$

  - Consider the following example. The current yield for a $15 - year$, 7% coupon bond with a par value of \$100 with a price of \$76.94 is given by:

$$Current\ Rate = \frac{\text{Annual Coupon}}{\text{Current Price}} = \frac{7}{76.94} = 9.1\%$$

  - The current yield has a number of drawbacks. It does not take into account the time value of money and it breaks down in the case of zero bonds.
- The bond yield
  - The bond yield γ is the discount rate that makes the present value of the cashflows equal to the market price of the bond. The bond yield is also known as the Internal Rate of Return or the Yield To Maturity or YTM for short. The bond yield satisfies the following equation:

$$P_{t_0} = \sum_{t=1}^{T} \frac{CF_t}{(1 + \frac{\gamma_{sa}}{m})^{t \cdot m}}$$

  - Consider the following example. Given the term structure below, determine the yield to maturity.

Maturity in years	Spot rate (ACT/ACT, cc, in %)
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

Figure 2.12: Term structure

First we determine the theoretical price of a two-year bond, providing a 6% coupon semi-annualy:

$$P_0 = \sum_{t=1}^{T} CF_t \cdot e^{-\gamma_c \cdot t}$$
$$P_0 = 3 \cdot e^{-0.050 \cdot 0.5} + 3 \cdot e^{-0.058 \cdot 1} + 3 \cdot e^{-0.064 \cdot 1.5} + 103 \cdot e^{-0.068 \cdot 2}$$
$$P_0 = 98.39$$

We can then determine the bond yield:

$$P_0 = \sum_{t=1}^{T} CF_t \cdot e^{-\gamma_c \cdot t}$$
$$98.39 = 3 \cdot e^{-\gamma \cdot 0.5} + 3 \cdot e^{-\gamma \cdot 1} + 3 \cdot e^{-\gamma \cdot 1.5} + 103 \cdot e^{-\gamma \cdot 2}$$

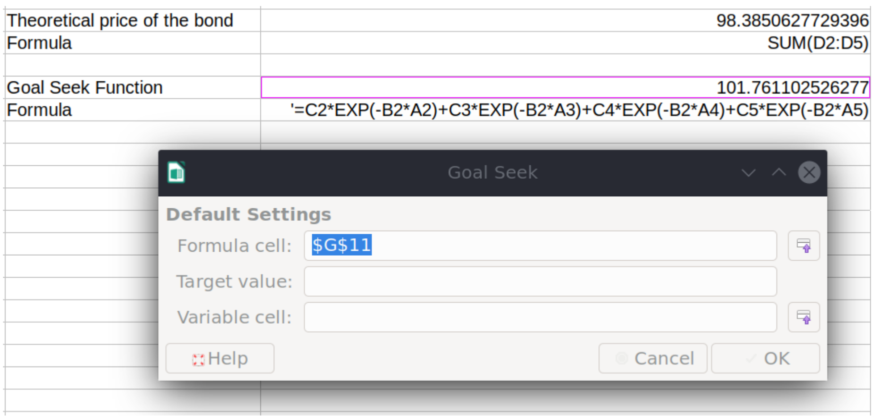
Solving the equation above manually is tricky, therefore we use numerical methods. The goal seek function in excel comes to the rescue. The table below gives us the term structure, cashflows and the present value of these cashflows.

	A	B	C	D
1	Maturity	Rates	CF	PV
2	0.5	5.00%	3	2.92592974
3	1	5.8%	3	2.83094984
4	1.5	6.4%	3	2.72539205
5	2	6.8%	103	89.9027911

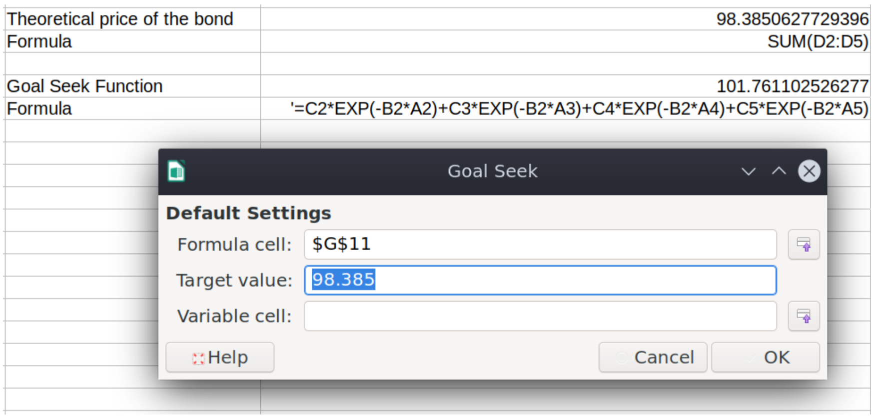
We calculated the price of the bond just before. This price is given in the first row. The second row displays the formula by which this theoretical price is obtained. The formula of the goal seeking function is displayed in the lats row. This is the formula we want sol solve numerically. In this case the cell $B2$ is our unknown and represents the unknown bond yield γ.

Theoretical price of the bond	98.3850627729396
Formula	SUM(D2:D5)
Goal Seek Function	101.761102526277
Formula	'=C2*EXP(-B2*A2)+C3*EXP(-B2*A3)+C4*EXP(-B2*A4)+C5*EXP(-B2*A5)

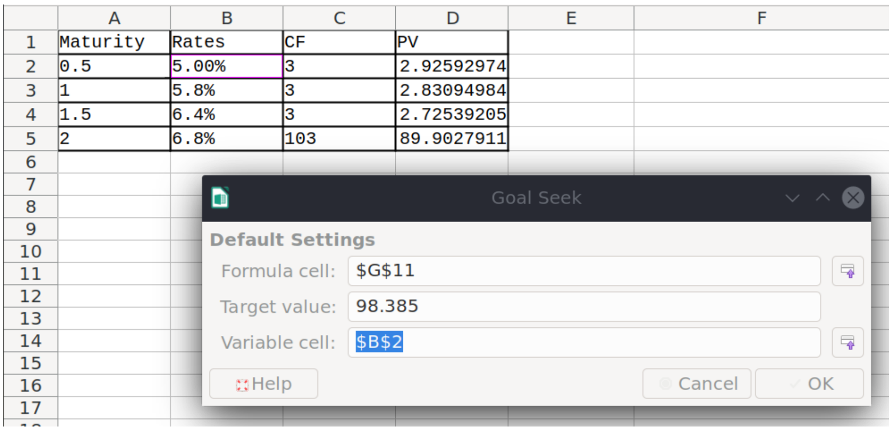
We select the cell with the formula we use for the goal seek.



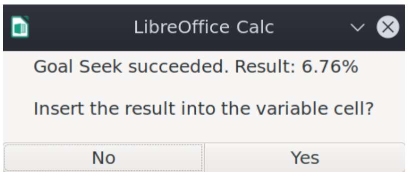
We select our target value i.e. the theoretical price of the bond.



We select the cell that represents our variable i.e. the unknown bond yield in the formula.



Our spreadsheet program shows the result.



- Yield for a portfolio of bonds
  - There are two possible approaches to calculate the yield for a portfolio of bonds:
    - ∗ Taking the weighted average of the yields of every bond, based on the market value of every bond.
    - ∗ Calculating the Internal Rate of Return for the whole portfolio.
  - We consider the following example.

Bond	Coupon	Maturity (in Years)	Par Value	Market Value	YTM	weights	w*YTM
A	7%	5	10000000	9208728.182	9%	16.08%	1.447%
B	10.50%	7	20000000	20000000	10.50%	34.93%	3.668%
C	6%	3	30000000	28050097.5	8.50%	48.99%	4.164%
			Total	57258825.68			**9.279%**

Figure 2.13: Yield for a portfolio of three bonds, calculated as the weighted average of the yields of the individual bonds.

Periods	Bond A	Bond B	Bond C	PortfolioCF	PV(PCF)
1	350000	1050000	900000	2300000	2195291
2	350000	1050000	900000	2300000	2095349
3	350000	1050000	900000	2300000	1999957
4	350000	1050000	900000	2300000	1908908
5	350000	1050000	900000	2300000	1822004
6	350000	1050000	30900000	32300000	24422395
7	350000	1050000		1400000	1010364
8	350000	1050000		1400000	964367
9	350000	1050000		1400000	920463.6
10	10350000	1050000		11400000	7153980
11		1050000		1050000	628921.5
12		1050000		1050000	600289.5
13		1050000		1050000	572961
14		21050000		21050000	10963574
					57258826
		YTM	4.770%		
		Difference	0		

Figure 2.14: Yield for a portfolio of three bonds, calculated as the IRR of the whole portfolio. For each period we determine the total cashflow that the portfolio generates. Each cashflow is the sum of all the coupons received in that period. We find the yield by equating the present value of these cashflows with the market value of the portfolio.

The coupon effect

- Suppose we have a flat term structure of interest rates i.e. the spot rate is the same for all maturities. This implies that the yield to maturity is equal to the spot rate. Consider for example the following two bonds:

$$110.69 = \frac{10}{1.06} + \frac{10}{1.06^2} + \frac{110}{1.06^3}$$
$$96.65 = \frac{4}{1.06} + \frac{4}{1.06^2} + \frac{104}{1.06^3}$$

- When we have a non-flat term structure, the yield to maturity of bonds with the same maturity depends on the height of the coupons of the bonds. Consider for example the following bonds:

$$105.84 = \frac{10}{1.04} + \frac{10}{1.06^2} + \frac{110}{1.08^3} \quad with\ \gamma = 7.745\%$$
$$89.96 = \frac{4}{1.04} + \frac{4}{1.06^2} + \frac{104}{1.08^3} \quad with\ \gamma = 7.886\%$$

  The yield to maturity of the first bond is 7.745%; the yield to maturity of the second bond is 7.886%. A bond with a higher coupon has a lower yield to maturity. This phenomenon is called the coupon effect. To quantify the coupon effect for bonds with the same credit risk and the same maturity, we need to:

  - Discount the cashflows of both bonds to determine their price and yield.
  - Take the difference between both yields.
  - This difference is the coupon effect.

  Under a normal spot rate curve, a coupon bond has a lower yield than a zero-coupon bond of equal maturity. Picking a zero-coupon bond over a coupon bond based purely on the higher yield to maturity of the zero-coupon bond is flawed. When we use the yield to maturity as a measure to compare bonds, we have to compare bonds with similar coupons.

Bond pricing in excel

- Functions:
  - PRICE-function. Returns the price per \$100 face value of a security that pays periodic interest.

$$= PRICE(settlement, maturity, rate, yld, redemption, \\ frequency, [basis])$$

  - COUPPCD-function. Returns the previous coupon date before the settlement date for a coupon bond.

$$= COUPPCD(settlement, maturity, frenquency, [basis])$$

  - COUPNCD-function. Returns the next coupon date after the settlement date.

$$= COUPNCD(settlement, maturity, frenquency, [basis])$$

  - COUPDAYBS-function. Returns the number of days from the beginning of the coupon period to the settlement date.

$$= COUPDAYBS(settlement, maturity, frenquency, [basis])$$

  - COUPDAYS-function. Returns the number of days in a coupon period that includes the settlement date.

$$= COUPDAYS(settlement, maturity, frenquency, [basis])$$

  - ACCRINT-function. Returns the accrued interest for a security that pays periodic interest.

$$= ACCRINT(id, fd, settlement, rate, par, frequency, [basis], [calc])$$

  - DOLLARDE-function. Converts a dollar price entered with a special notation to a dollar price displayed as a decimal number. The DOLLARFR function does the opposite conversion.

$$= DOLLARDE(fractional_dollar, fraction)$$

- Arguments:
  - **Settlement**. The security's settlement date: the day afther the issue date: the day afther the security was acquired by the buyer.
  - **maturity**. The maturity date: the day the security expires.
  - **Rate** The security's annual coupon rate.
  - **Yld**. The security's annual yield.
  - **Redemption**. The redemption value: the price at which the issuing company may choose to repurchase a security before its maturity date per $100 face value.
  - **Frequency**. The number of coupon payments per year.
  - **Id**. The issue date of the security.
  - **Fd**. The first interest date of security.
  - **Par**. The par value of security.
  - **Basis**. The day count basis used.
  - **Calc**. The calculation method.
  - **Fractional**_dollar. The dollar component in special fractional notation.
  - **Fraction**: The denominator in the fractional unit. $8 = 1/8, 16 = 1/16, 32 = 1/32$, etc.
- Remark. Dates have to entered using the date function.

- Consider the following example. We are given:
  - The settlement date
  - The issue date
  - The maturity date
  - The face value
  - The coupon rate
  - The coupon frequency
  - The day count

  We can then determine:
  - The next coupon date i.e. the coupon date that first follows the settlement date:
    $$Next\ coupon\ date = COUPNCD$$
  - The last coupon date i.e. the coupon date that first precedes the settlement date:
    $$Last\ coupon\ date = COUPPCD$$
  - The number of days accrued:
    $$\begin{aligned} Number\ of\ days\ accrued &= Next\ coupon\ date - Settlement\ date \\ &= COUPDAYBS \end{aligned}$$
  - The number of days between the last and the next coupon date:
    $$\begin{aligned} Number\ of\ days\ between &= Next\ coupon\ date - Last\ coupon\ date \\ &= COUPDAYS \end{aligned}$$
  - The accrued interest:
    $$\begin{aligned} Accrued\ interest &= \frac{Number\ of\ days\ accrued}{Number\ of\ days\ between} * \frac{Coupon\ rate}{Frequency} * Face\ Value \\ &= ACCRINT \end{aligned}$$

	A	B	C	D
1	**Pricing a Bond in Excel**			
2				
3	Settlement date	27/06/2008		
4	Issue date	15/05/2008		
5	Maturity date	15/05/2018		
6	Face Value	100		
7				
8	Coupon	3.875%		
9	Frequency	2		
10	Day count	1		
11	First coupon date	15/11/2008		
12	Last coupon date (LCD)	15/05/2008		=COUPPCD(B3;B5;B9;B10)
13	Next coupon date (NCD)	15/11/2008		=COUPNCD(B3;B5;2;1)
14				
15	Number of days accrued (LCD-SD)	43		=COUPDAYBS(B3;B5;B9;B10)
16	Number of days between LCD-NCD	184		=COUPDAYS(B3;B5;B9;B10)
17	Accrued interest	0.452785326		=B6*(B8/B9)*(B15/B16)
18	or	0.452785326		=ACCRINT(B4;B11;B3;B8;B6;B9;B10;1)
19				
20	Quoted price (32nds)	98.22		
21	Clean price (dec)	98.687500		=DOLLARDE(B20;32)
22				
23	Dirty price (dec)	99.140285		=B21+B17
24				
25	Yield to Maturity	4.03694%		<= determined by goal seek
26				
27	Yield to Maturity	4.03694%		=YIELD(B3;B5;B8;B21;B6;2;1)
28	Clean Price (dec)	98.687500		=PRICE(B3;B5;B8;B27;B6;2;1)
29				

Figure 2.15: Example of bond pricing in excel.

	G	H	I	J	K	L	M
1							
2		Timing	Cash Flow	Time to payment (Semi-annual)	PV(CF)		
3	1	15/11/2008	1.9375	0.766	€ 1.908		=(H3-$B$3)/(H3-$B$4)
4	2	15/05/2009	1.9375	1.766	€ 1.870		=1+J3
5	3	15/11/2009	1.9375	2.766	€ 1.833		
6	4	15/05/2010	1.9375	3.766	€ 1.797		
7	5	15/11/2010	1.9375	4.766	€ 1.761		
8	6	15/05/2011	1.9375	5.766	€ 1.727		
9	7	15/11/2011	1.9375	6.766	€ 1.692		
10	8	15/05/2012	1.9375	7.766	€ 1.659		
11	9	15/11/2012	1.9375	8.766	€ 1.626		
12	10	15/05/2013	1.9375	9.766	€ 1.594		
13	11	15/11/2013	1.9375	10.766	€ 1.562		
14	12	15/05/2014	1.9375	11.766	€ 1.532		
15	13	15/11/2014	1.9375	12.766	€ 1.501		
16	14	15/05/2015	1.9375	13.766	€ 1.472		
17	15	15/11/2015	1.9375	14.766	€ 1.442		
18	16	15/05/2016	1.9375	15.766	€ 1.414		
19	17	15/11/2016	1.9375	16.766	€ 1.386		
20	18	15/05/2017	1.9375	17.766	€ 1.358		
21	19	15/11/2017	1.9375	18.766	€ 1.332		
22	20	15/05/2018	101.9375	19.766	€ 68.673	=PV($B$25/2;J22;;-I22)	
23							
24					€ 99.140285	=SUM(K3:K23)	

Figure 2.16: Example of bond pricing in excel.

Par bonds and non-par bonds

- Par bonds. A par bond refers to a bond that currently trades at its face value. A par bond comes with a coupon rate that is identical to its yield to maturity. Mathematically:

$$Price = Face\,Value$$
$$C = F * \gamma$$
$$c = \gamma$$

  We can easily proof that the bond trades at its face value:

$$\begin{aligned} P_{t_0} &= C \cdot \left[\frac{1 - \frac{1}{(1+\gamma)^T}}{\gamma}\right] + \frac{F}{(1+\gamma)^T} \\ &= F \cdot \gamma \cdot \left[\frac{1 - \frac{1}{(1+\gamma)^T}}{\gamma}\right] + \frac{F}{(1+\gamma)^T} \\ &= F \cdot \left[1 - \frac{1}{(1+\gamma)^T} + \frac{F}{(1+\gamma)^T} \right. \\ &= F \end{aligned}$$

  As we see: a bond that comes with a coupon rate that is identical to the market interest rate, is always a par bond.

- Non-par bonds. Here, we make the distinction between:

  1. Premium bonds. These are bonds that sell at a premium with regards to their face value and where the coupon rate is greater than the yield to maturity. The premium will be lost over the lifetime of the bond. Mathematically:

$$P > F \; and \; c > \gamma$$

  2. Discount bonds. These are bonds that sell at a discount with regards to their face value and where the coupon rate is smaller than the yield to maturity. The discount is earned during the lifetime of the bond. Mathematically:

$$P < F \; and \; c < \gamma$$

- We can illustrate how the market price of a premium bond and a discount bond evolve over their lifetimes graphically:

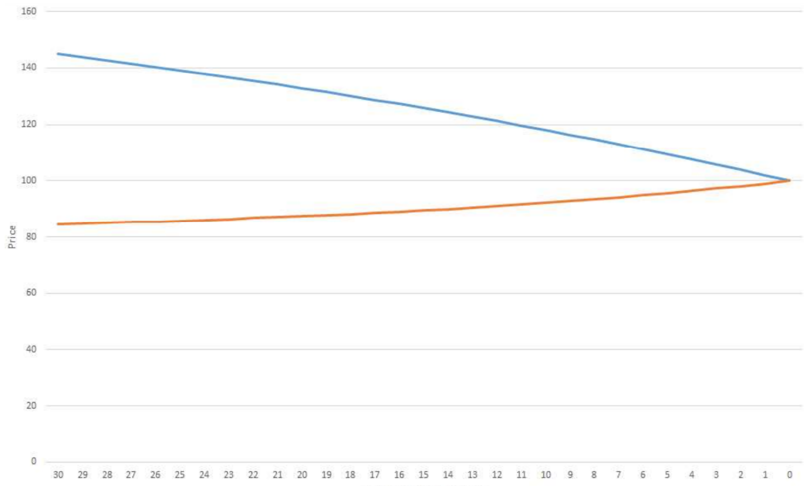


Figure 2.17: Evolution of market values of a premium bond (blue) and a discount bond (orange).

We can see that the price of the bond evolves towards the face value of the bond. We call this effect the pull to par.

- We can therefore decompose price changes in bonds into two factors:
  - Changes in time to maturity id est the pull to par effect.
  - Changes in yield: the market effect id est changes in the price of a bond due to changes in the demand or the supply for the bond.

Sources of a bond's return

- The return of a bond can be decomposed into different factors. These factors are:
  - Periodic coupon payments.
  - Income from the reinvestment of the coupons that are received.
  - Capital gains or losses as the bond matures, is called or is sold.
- Different measurements for the bonds return take one or more of these factors into account. Some of these measurements are:
- The coupon rate c which does not provide us with sensible information:

$$c = \frac{C}{F}$$

- The current yield CY_t which only takes the periodic coupons into account:

$$CY_t = \frac{C}{V_t}$$

- The capital gains rate CG_t which gives the percentage change in value over one period, ignoring the coupon:

$$CG_t = \frac{V_{t+1} - V_t}{V_t}$$

- The total return R_t which is the sum of the current yield and the capital gain

$$R_t = CY_t + CG_t$$

- The yield to maturity which takes all factors into account. However it is only a promised" yield. Id est the yield to maturity will only be realized if:
  - The bond is held to maturity.
  - The coupons are reinvested at the yield to maturity.

The price-yield relationship

- We can illustrate the relationship between the price and the yield to maturity of a bond graphically.

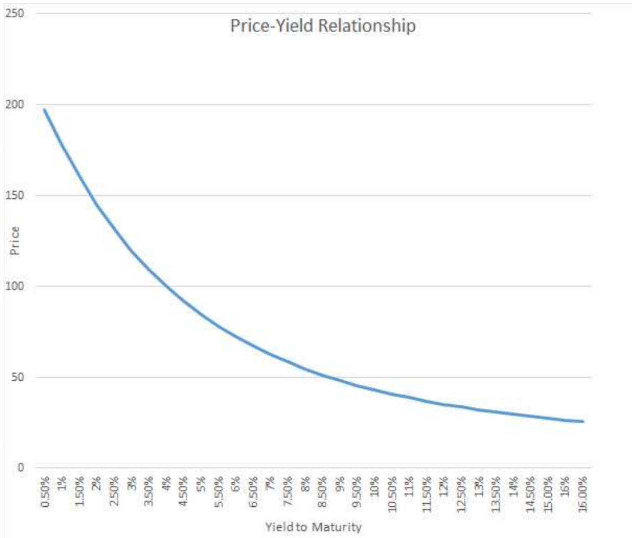


Figure 2.18: Relationship between the yield to maturity and the price of a bond. A higher price corresponds to a lower yield to maturity.

- The relationship between the price and the yield to maturity for two bonds with different coupons.

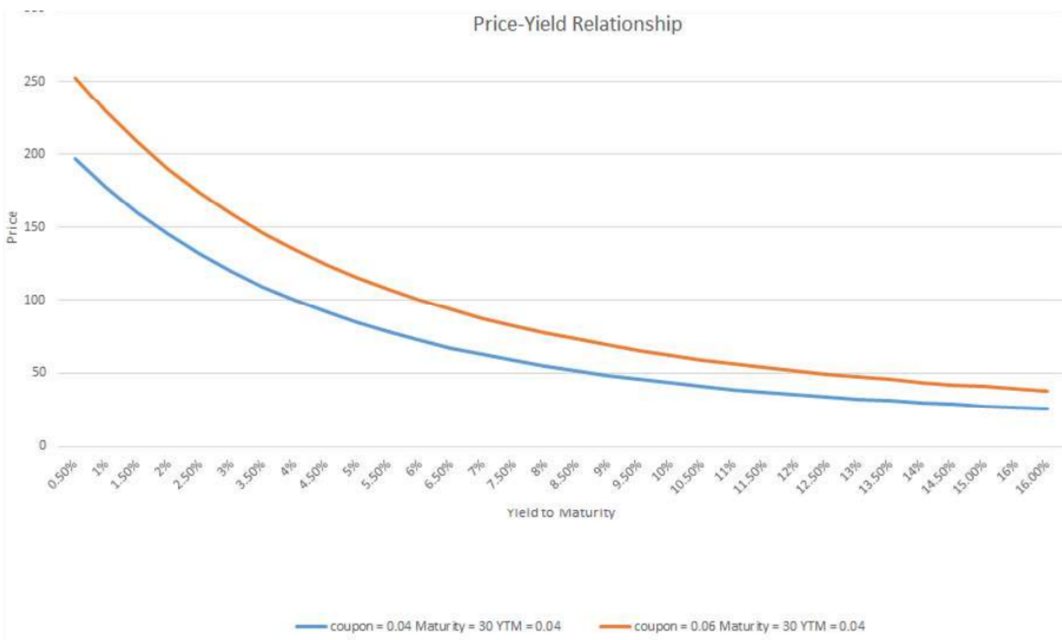


Figure 2.19: Relationship between the yield to maturity and the price for two bonds with different coupons. A higher coupon bond has a higher yield to maturity when the prices are equal. A higher coupon bond has a higher price, when the yield to maturity of both bonds is equal.

- The relationship between the price and the yield to maturity for two bonds where the coupons are reinvested at different yields.

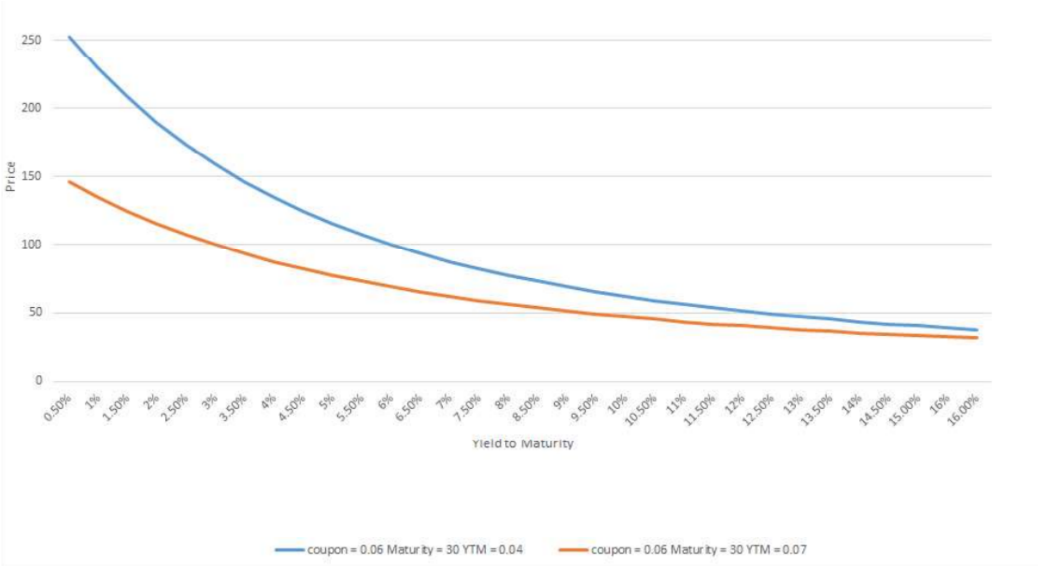


Figure 2.20: Reinvesting the coupons at a higher yield grants a higher price. The difference becomes smaller as maturity closes in. A bond where the coupons can be reinvested at a higher rate, has a YTM that is less sensitive to price changes of the bond.

- The relationship between the price and the yield to maturity for two bonds with a different time to maturity.

Figure 2.21: A bond with a long time to maturity has a yield that is more sensitive to price changes than an equal bond with a small time to maturity.

Price sensitivity

- Recall the following relationship for a bond:

$$P_{t_0} = \sum_{t=1}^{T} \frac{CF_t}{(1+\gamma)^t}$$

We can split up the equation into two different parts:

1. The sum of the present value of each coupon.
2. The present value of the nominal value or the face value of the bond.

The equation then becomes:

$$\begin{aligned} P_{t_0} &= \sum_{t=1}^{T} (\frac{C_t}{(1+\gamma)^t}) + \frac{F}{(1+\gamma)^T} \\ &\Big| \sum_{k=0}^{n-1} a \cdot r^k = a \cdot \frac{1-r^n}{1-r} \\ &= C \cdot \left[\frac{1 - \frac{1}{(1+\gamma)^T}}{\gamma}\right] + \frac{F}{(1+\gamma)^T} \end{aligned}$$

- Hence:

$$P = f(c, T, \gamma)$$

With

  - c: the coupon rate.
  - γ: the yield to maturity.
  - F: the face value.

- We can measure yield sensitivity of the price by calculating the price change if the yield changes with one basis point. This quantity is called the dollar value of a basis point (DVBP) or the price value of a basis point (PVBP) or the basis point value (BPV). Mathematically:

$$\begin{aligned} BPV^+ &= P(\gamma, c, T) - P(\gamma + 1bp, c, T) \\ BPV^- &= P(\gamma, c, T) - P(\gamma - 1bp, c, T) \\ BPV^{+/-} &= P(\gamma - 0.5bp, c, T) - P(\gamma - 0.5bp, c, T) \end{aligned}$$

- A basis point is defined as:

$$BP = 0.01\% = 0.0001 = \frac{1}{10000}$$

- Conclusions:
  - Bond prices and yields are inversely related.
  - For small changes in the yield, the price impact is symmetric.
  - Negative yield shocks have a stronger impact than positive yield shocks.
  - Bonds with:
    * low coupons,
    * long maturities,
    * low yields,

    show most price variability id est they carry most interest rate risk.

Bond price prediction after a yield shock

- We want to predict the expected bond price after a yield shock.
- A basis point value or BPV denotes the change in the price of a bond given a basis point change in the yield of the bond. Given this definition, the following must hold:

$$P_{new} = P_{old} - BPV \cdot \Delta\gamma$$
$$\Delta P = -BPV \cdot \Delta\gamma$$

If we set $\Delta\gamma = 1$ and divide both sides of the equation by P_{old}, we get the following expression:

$$\frac{P_{new} - P_{old}}{P_{old}} = -\frac{BPV}{P_{old}}$$

The ratio $\frac{BPV}{P}$ gives the negative percentual change in price for a $1bp$ shock in the yield γ.

We can illustrate the relationship graphically:

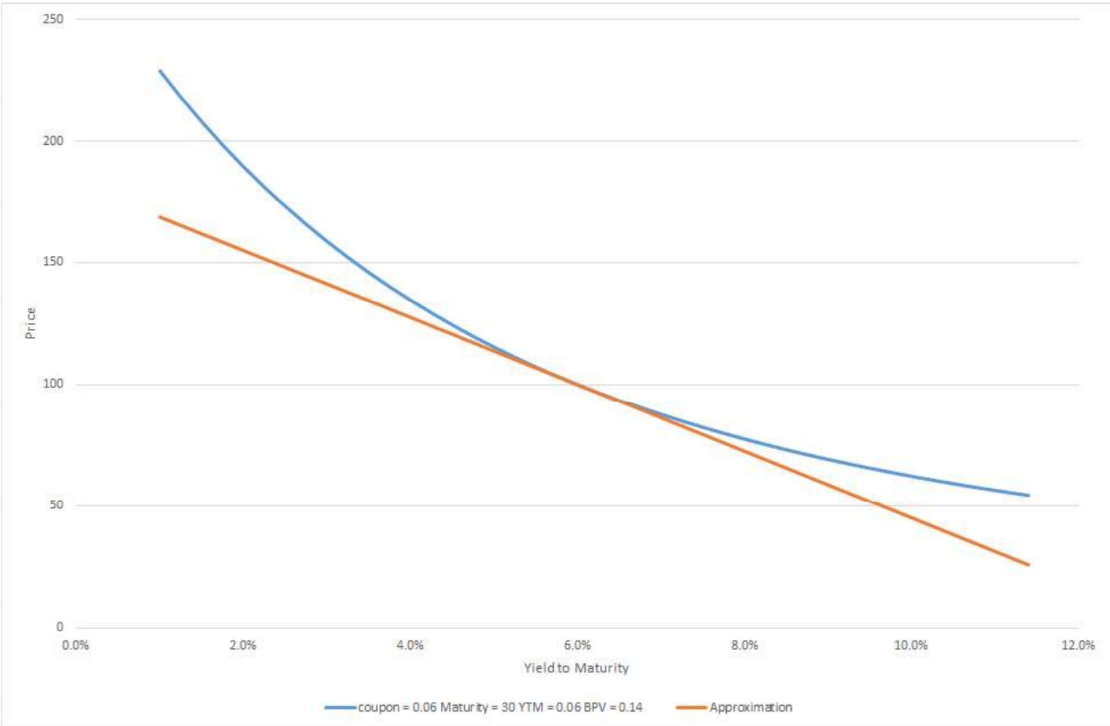


Figure 2.22: An approximation based on the BPV of how the price changes with the yield to maturity.

The price-yield relationship formalized

- The duration of a financial asset that consists of fixed cashflows, such as a bond, is the weighted average of the times until those fixed cashflows are received.
- The Macaulay duration is defined as the weighted average maturity of cashflows, in which the time of receipt of each payment is weighted by the present value of that payment:

$$D_{mac} = \frac{\sum_{t=1}^{T} t \cdot PV(C_t)}{\sum_{t=1}^{T} PV(C_t)}$$

- The modified duration is defined as the the Macaulay duration after discounting for one compounding period:

$$D_{mod} = \frac{D_{mac}}{1 + \frac{\gamma}{m}}$$

- We are interested in how the price of a bond changes when the yield of that bond changes. The infinitesimal change in the value of a mathematical function can be estimated by taking the first derivative of that function. We know that:

$$P = \sum_{t=1}^{T} \frac{CF_t}{(1+\gamma)^t}$$

We can take the first derivative of the price with regards to the yield:

$$\frac{dP}{d\gamma} = -\frac{1}{1+\gamma}\left[\frac{1 \cdot CF_1}{(1+\gamma)^1} + \frac{2 \cdot CF_2}{(1+\gamma)^2} + ... + \frac{T \cdot CF_T}{(1+\gamma)^T}\right]$$

If we rearrange the equation, we get a relationship between dP and $d\gamma$:

$$dP = -\frac{1}{1+\gamma}\left[\frac{1 \cdot CF_1}{(1+\gamma)^1} + \frac{2 \cdot CF_2}{(1+\gamma)^2} + ... + \frac{T \cdot CF_T}{(1+\gamma)^T}\right] \cdot d\gamma$$

The absolute change in the price of the bond equals the negative of the dollar duration times the change in the yield to maturity.

- But what about relative changes? We know that:

$$dP = -\frac{1}{1+\gamma}\Big[\frac{1 \cdot CF_1}{(1+\gamma)^1} + \frac{2 \cdot CF_2}{(1+\gamma)^2} + ... + \frac{T \cdot CF_T}{(1+\gamma)^T}\Big] \cdot d\gamma$$

It must follow that:

$$\begin{aligned}
\frac{dP}{P} &= -\frac{1}{1+\gamma}\Big[\frac{1 \cdot CF_1}{(1+\gamma)^1} + \frac{2 \cdot CF_2}{(1+\gamma)^2} + ... + \frac{T \cdot CF_T}{(1+\gamma)^T}\Big] \cdot d\gamma \cdot \frac{1}{P} \\
\frac{dP}{P} &= -\frac{1}{1+\gamma}\Big[\Big(\frac{\frac{CF_1}{(1+\gamma)^1}}{P}\Big) \cdot 1 + \Big(\frac{\frac{CF_2}{(1+\gamma)^2}}{P}\Big) \cdot 2 + ... + \Big(\frac{\frac{CF_T}{(1+\gamma)^T}}{P}\Big) \cdot T\Big] \cdot d\gamma \\
\frac{dP}{P} &= -\frac{1}{1+\gamma} \cdot [D^{ac}_{mac}] \cdot d\gamma
\end{aligned}$$

- We can also derive this relationship for the case of periodic compounding:

$$P = \sum_{t=1}^{m \cdot T} \frac{CF_t}{(1+\frac{\gamma}{m})^t}$$

$$\begin{aligned}
\frac{d}{d\gamma}\Big[\frac{CF_t}{(1+\frac{\gamma}{m})^{-t}}\Big] &= \frac{d}{d\gamma}\Big[CF_t \cdot (1+\frac{\gamma}{m})\Big] = CF_t \cdot -t \cdot (1+\frac{\gamma}{m})^{-t-1} \cdot \frac{1}{m} + 0 \\
&= \frac{CF_t}{(1+\frac{\gamma}{m})^t} \cdot \frac{1}{m} \cdot -t \cdot \frac{1}{1+\frac{\gamma}{m}} \\
\frac{dP}{d\gamma} &= -\frac{1}{m} \cdot \frac{1}{1+\frac{\gamma}{2}}\Big[\frac{1 \cdot CF_1}{(1+\frac{\gamma}{m})^1} + \frac{2 \cdot CF_2}{(1+\frac{\gamma}{m})^2} + ... + \frac{mT \cdot CF_{mT}}{(1+\frac{\gamma}{m})^{mT}}\Big] \\
\frac{dP}{P} &= -\frac{1}{m} \cdot \frac{1}{1+\frac{\gamma}{2}}\Big[\frac{1 \cdot CF_1}{(1+\frac{\gamma}{m})^1} + \frac{2 \cdot CF_2}{(1+\frac{\gamma}{m})^2} + ... + \frac{mT \cdot CF_{mT}}{(1+\frac{\gamma}{m})^{mT}}\Big] \cdot \frac{1}{P} \cdot d\gamma \\
\frac{dP}{P} &= -\frac{1}{1+\frac{\gamma}{m}}\Big[\Big(\frac{\frac{CF_1}{(1+\frac{\gamma}{m})^1}}{P}\Big) \cdot 0.5 + \Big(\frac{\frac{CF_2}{(1+\frac{\gamma}{m})^2}}{P}\Big) \cdot 1 + ... + \Big(\frac{\frac{CF_{2T}}{(1+\frac{\gamma}{m})^{2T}}}{P}\Big)\Big] \cdot d\gamma \\
\frac{dP}{P} &= -\frac{1}{1+\frac{\gamma}{m}}\big[D^{(sac)}_{mac}\big] d\gamma
\end{aligned}$$

- We can also derive this relationship for the case of continuous compounding:

$$P = \sum_{t=1}^{T} CF_t \cdot e^{-t \cdot \gamma}$$

$$\begin{aligned}
\frac{dP}{d\gamma} &= -\sum_{t=1}^{T} t \cdot CF_t \cdot e^{-t \cdot \gamma} \\
\frac{dP}{P} &= \Big[-\sum_{t=1}^{T} t * CF_t \cdot e^{-t \cdot \gamma} \Big] \cdot \frac{1}{P} \cdot d\gamma \\
\frac{dP}{P} &= -\Big[\Big(\frac{CF_1 \cdot e^{-1 \cdot \gamma}}{P} \Big) \cdot 1 + \Big(\frac{CF_2 \cdot e^{-2 \cdot \gamma}}{P} \Big) \cdot 2 + ... + \Big(\frac{CF_T \cdot e^{-T \cdot \gamma}}{P} \Big) \cdot T \Big] \cdot d\gamma \\
\frac{dP}{P} &= -[D^{(cc)}] d\gamma
\end{aligned}$$

Approximating the price-yield relation with a Taylor-expansion

- First order expansion A Taylor expansion of the first order is given by:

$$f(x_0 + \Delta x) = f(x_0) + \frac{df}{dx_0}\Delta x + ...$$

  If we apply this formula to the relationship between the price P and the yield γ $P = f(\gamma)$, we get:

$$P_{new} = f(\gamma + \Delta\gamma) = P_0 + [-D_{mod} \cdot P_0]\Delta\gamma + ...$$
$$P_{new} \approx P_0 + [-D_{\$}]\Delta\gamma$$

- Second order expansion A Taylor expansion of the second order is given by:

$$f(x_0 + \Delta x) = f(x_0) + \frac{df}{dx_0}\Delta x + \frac{1}{2!}\frac{d^2 f}{dx_0^2}\Delta x^2 + ...$$

  If we apply this formula to the relationship between the price P and the yield γ: $P = f(\gamma)$, we get:

$$P_{new} = f(\gamma + \Delta\gamma) = P_0 + [-D_{mod} \cdot P_0]\Delta\gamma + \frac{1}{2} \cdot [CVX \cdot P_0] \cdot \Delta\gamma^2 + ...$$
$$P_{new} \approx P_0 - [D_{\$}]\Delta\gamma + \frac{1}{2} \cdot CVX_{\$}\Delta\gamma^2$$

  With:

$$CVX_{\$} = CVX \cdot P_0$$

Calculating the duration in Excel

Example:

	F	G	H	I	J	K	L	M
27		Timing		Cash Flow	Time to payment (Semi-annual)	PV(CF)		
28		1	15/11/2008	1.9375	0.766	€ 1.908	0.0147	=K28*J28/$K$49
29		2	15/05/2009	1.9375	1.766	€ 1.870	0.0333	
30		3	15/11/2009	1.9375	2.766	€ 1.833	0.0512	
31		4	15/05/2010	1.9375	3.766	€ 1.797	0.0683	
32		5	15/11/2010	1.9375	4.766	€ 1.761	0.0847	
33		6	15/05/2011	1.9375	5.766	€ 1.727	0.1004	
34		7	15/11/2011	1.9375	6.766	€ 1.692	0.1155	
35		8	15/05/2012	1.9375	7.766	€ 1.659	0.1300	
36		9	15/11/2012	1.9375	8.766	€ 1.626	0.1438	
37		10	15/05/2013	1.9375	9.766	€ 1.594	0.1570	
38		11	15/11/2013	1.9375	10.766	€ 1.562	0.1697	
39		12	15/05/2014	1.9375	11.766	€ 1.532	0.1818	
40		13	15/11/2014	1.9375	12.766	€ 1.501	0.1933	
41		14	15/05/2015	1.9375	13.766	€ 1.472	0.2043	
42		15	15/11/2015	1.9375	14.766	€ 1.442	0.2148	
43		16	15/05/2016	1.9375	15.766	€ 1.414	0.2248	
44		17	15/11/2016	1.9375	16.766	€ 1.386	0.2344	
45		18	15/05/2017	1.9375	17.766	€ 1.358	0.2434	
46		19	15/11/2017	1.9375	18.766	€ 1.332	0.2521	
47		20	15/05/2018	101.9375	19.766	€ 68.673	13.6918	
48								
49						€ 99.140285	16.51	=SUM(L28:L48)
50						Macauley Duration	8.25	=L49/2

Figure 2.23: Example of calculating the duration in Excel

Duration of a portfolio

The duration of a portfolio is the weighted average of the duration of the individual assets.

The duration of a perpetual

The modified duration of a perpetual equals the reciprocal of the yield:

$$\begin{aligned} P &= \frac{c \cdot 100}{\gamma} \\ \frac{dp}{d\gamma} &= -\frac{c \cdot 100}{\gamma^2} \\ \frac{dp}{d\gamma} \cdot \frac{1}{P} &= -\frac{c \cdot 100}{\gamma^2} \cdot \frac{1}{P} \\ \frac{dP}{P} &= -\frac{1}{\gamma} d\gamma \end{aligned}$$

3 Forward contracts

3.1 Introduction

A forward contract is an agreement concluded between two counterparties in the OTC market in which they agree to buy or sell an underlying asset, at a certain point in time in the future, at a predetermined price.

There are two parties involved in a forward contract:

- The long party. The long party has the obligation to buy the underlying at the maturity of the forward contract at the contracted delivery price. The long party benefits when the price of the underlying increases.
- The short party. The short party has the obligation to sell the underlying at the maturity of the forward contract at the contracted delivery price. The short party benefits when the price of the underlying decreases.

The forward price is the delivery price for which the contract is of no value for both parties at the inception of the forward contract. The forward price is thus the arbitrage-free delivery price. Arbitrage is not possible when the delivery price is equal to the forward price.

Throughout this chapter we will use the following symbols:

- Delivery price: DP
- Maturity data: T
- Spot price at time t: S_t
- Payoff at maturity: S_T
- Forward price: F

3.2 Payoff at maturity

At maturity, the long pays the contracted delivery price DP and receives the underlying. The value of the underlying at maturity is the spot price of the underlying at that point in time s_T. The long can realize an immediate positive cashflow by selling the underlying spot. In that case, he receives a positive cashflow of magnitude S_T The payoff at maturity for the long is thus given by:

$$f_T^{long} = S_T - DP$$

The short receives the contracted delivery price DP when selling the underlying. However, the value of the underlying is equal to the spot price at that point in time S_T. The payoff at maturity for the short is therefore equal to:

$$f_T^{short} = DP - S_T$$

3.3 Determining the forward price

3.3.1 Introductory example

- Suppose you go shopping for a painting. You could ask a friend to buy the painting and conclude forward agreement at the same time.
- To finance the purchase of the painting, your friend has to take out a loan with an interest rate r. The forward price has to reflect this financing cost. This is done by compounding the spot price of the underlying in the forward contract using this interest rate r:
$$F = S_{t_0} \cdot (1 + r)^\tau$$
- Your friend plans to organize an exhibition where he shows your painting. This means the underlying of the forward contract is generating income. In doing so your friend recovers part of the financing cost. This income also has to be taken into account into the calculation of the forward price. The income is usually expressed in terms of a discount rate δ:
$$F = S_{t_0} \cdot \frac{1}{(1 + \delta)^\tau}$$

3.3.2 Generalization

In general the following equation holds true:

$$F = S_{t_0} \cdot \frac{(1 + r)^\tau}{(1 + \delta)^\tau}$$

The income that is generated by the underlying during the lifespan of the forward contract is expressed by discounting the spot price of the underlying at the time of the inception of the contract. The costs that are generated by the underlying during the lifespan of the forward contract can be expressed by compounding the spot price of the underlying at the time of the inception of the contract. Sometimes, there are costs associated with storing or holding an asset. These costs are called holding costs or costs of carry. They are the costs of carrying the asset over the lifespan of the forward contract.

3.3.3 The replicating portfolio

- A portfolio that produces the same cashflows as a financial instrument, in all possible states of the world is called a replicating portfolio for that financial instrument. When this is the case, the price of the replicating portfolio has to be equal to the price of the financial instrument. If this is not the case, arbitrage opportunities arise.
- The payoff of a derivative is determined by the value of the underlying asset. It is therefore very likely that we can construct a replicating portfolio for most financial derivatives. Pricing derivatives is based upon the idea of the replicating portfolio.

3.3.4 Categories of underlying asset

Financial derivatives have a plethora of possible underlying assets. We make the distinction between:

- Investment assets. These assets are held by a significant number of people, solely for investment purposes. These assets are bought and sold without many restrictions.
- Consumption assets. These assets are held primarily for consumption purposes. There are some restrictions on buying and selling these assets.

In order to do arbitrage, one needs to be able to take long and short positions in the underlying. Going short in a consumption asset can prove difficult. This is because these assets have applications outside of investment and finance. For example, some consumption assets are used in production processes. Owners of these consumption assets might be reluctant to enter into short selling constructions.

3.3.5 Replicating a long forward contract

- Consider a forward contract on a non-dividend-paying investment asset.
- There are no cashflows between the time of the inception of the forward contract t_0 and maturity T. There is a single cashflow at maturity T which is equal to:

$$f_T^{long} = S_T - DP$$

- To set up a replicating portfolio, one needs to replicate the cashflows of the long forward under all possible circumstances. Setting up a replicating portfolio for a long forward contract would involve the following steps:
  - Buying the underlying asset spot at time t_0. This involves a negative cashflow equal to S_{t_0}.
  - Borrowing the present value of the delivery price DP. This involves a positive cashflow equal to $PV(DP)$.
  - Selling the underlying asset spot at maturity T. This involves a positive cashflow equal to S_T.
  - Repaying the borrowed amount plus interest. This negative involves a cashflow equal to DP.
- The total cost of the replicating portfolio is equal to the sum of all cashflows at time t_0:

$$-S_{t_0} + PV(DP)$$

  - The spot price of the underlying at time t_0 of the inception of the forward contract S_{t_0}, will evolve to the spot price of the underlying at maturity S_T.
  - The borrowed amount $PV(DP)$ will accrue to DP at maturity T.
- The payoff at maturity is equal to the sum of all cashflows at that time. The payoff at maturity is thus given by:

$$f_T^{repl.\,port.} = S_T - DP$$

- We can easily see that the payoff of the replicating portfolio is equal to the payoff of the long forward contract:

$$f_T^{repl.\,port.} = f_T^{long\,fw.}$$

  It must therefore be true that the price of the replicating portfolio is equal to the price of entering the long forward contract:

$$P^{repl.\,port.} = P^{long\,fw.}$$

  We know however that a forward contract is of no value to both parties at the inception of the contract. There is therefore no cost associated with entering into a forward contract:

$$P^{long\,fw.} = 0$$

The following must therefore be true:

$$\begin{aligned} -S_{t_0} + PV(DP) &= 0 \\ -S_{t_0} &= PV(DP) \\ DP &= S_{t_0} \cdot (1+r) \\ F &= S_{t_0} \cdot (1+r) \end{aligned}$$

The delivery price has to be equal to the future value of S_{t_0} in order for the condition f_{t_0} to hold. This delivery price $DP = S_{t_0}$ is called the forward price or the arbitrage-free delivery price F.

3.3.6 Replicating a short forward contract

- Consider a forward contract on a non-dividend-paying investment asset.
- There are no cashflows between the time of the inception of the forward contract t_0 and maturity T. There is a single cashflow at maturity T which is equal to:

$$f_T^{long} = DP - S_T$$

- To set up a replicating portfolio, one needs to replicate the cashflows of the short forward under all possible circumstances. Setting up a replicating portfolio for a short forward contract would involve the following steps:
  - Shorting the underlying asset. This involves a positive cashflow of magnitude S_{t_0}.
  - Investing the present value of the delivery price. This involves a negative cashflow of magnitude $PV(DP)$.
  - Buying the underlying asset back at maturity T. This involves a negative cashflow of magnitude S_T.
  - Receiving the proceeds of the investment. This involves a positive payoff with magnitude DP.
- The total cost of the replicating portfolio is equal to the sum of all cashflows at time t_0:

$$S_{t_0} - PV(DP)$$

  - The spot price of the underlying at time t_0 of the inception of the forward contract S_{t_0}, will evolve to the spot price of the underlying at maturity S_T.
  - The invested amount $PV(DP)$ will accrue to DP at maturity T.
- The payoff at maturity is equal to the sum of all cashflows at that time. The payoff at maturity is thus given by:

$$f_T^{repl.\ port.} = DP - S_T$$

- We can easily see that the payoff of the replicating portfolio is equal to the payoff of the short forward contract:

$$f_T^{repl.\,port.} = f_T^{short\;fw.}$$

It must therefore be true that the price of the replicating portfolio is equal to the price of entering the short forward contract:

$$P^{\,repl.\,port.} = P^{\,short\;fw.}$$

We know however that a forward contract is of no value to both parties at the inception of the contract. There is therefore no cost associated with entering into a forward contract:

$$P^{\,short\;fw.} = 0$$

The following must therefore be true:

$$\begin{aligned} S_{t_0} - PV(DP) &= 0 \\ S_{t_0} &= PV(DP) \\ DP &= S_{t_0} \cdot (1+r) \\ F &= S_{t_0} \cdot (1+r) \end{aligned}$$

The delivery price has to be equal to the future value of S_{t_0} in order for the condition f_{t_0} to hold. This delivery price $DP = S_{t_0}$ is called the forward price or the arbitrage-free delivery price F.

3.3.7 Intermezzo: short selling

- Short selling involves selling an asset that you do not own.
- Short selling an asset can be done trough security lending or trough a reverse repo.
- Sometimes, a lending fee is charged to shorter.
- The shorter has to pay back any income that would be received on the shorted assets in normal circumstances.
- The shorter has to maintain a margin account with his broker.
- Short selling is usually only possible for investment assets.

3.3.8 Summary: forward price of a non-dividend paying investment grade asset.

- The forward price of a non-dividend paying investment grade asset is derived by setting up a replicating portfolio and equating the price of the replicating portfolio to the price of the forward contract. The forward price is the arbitrage free delivery price.

– The forward price is given by:

$$F = S_{t_0} \cdot (1 + r\tau)$$
$$F = S_{t_0} \cdot (1 + \frac{r}{m})^{\tau m}$$
$$F = S_{t_0} \cdot e^{r\tau}$$

3.3.9 Example

- Consider a three-month forward contract on an underlying asset with a unit price of \$100. The risk-free interest rate is equal to 5% p.a. c.c.
- Determine the delivery price DP for this three-month, zero-cost forward contract.
- The forward price F in the case of continuous compounding is given by:

$$\begin{aligned} F &= S_{t_0} \cdot e^{r\tau} \\ &= 100 \cdot e^{0.05 \cdot 0.25} \\ &= 101.26 \end{aligned}$$

- We can easily verify this result by calculating the value of the forward contract at time t_0. The value of the forward contract f_{t_0} is given by:

$$\begin{aligned} f_{t_0} &= -S_t + PV(DP) \\ &= -100 + 101.26 \cdot e^{-0.05 \cdot 0.25} \\ &= 0 \end{aligned}$$

- This result agrees with our assumption that the forward is of no value at the inception of the contract.

3.3.10 Arbitrage opportunities

We already stated that the forward price is also called the arbitrage-free delivery price. This is because when the delivery price is out of line with the forward price $DP \neq F$, arbitrage opportunities arise. Id est it is then possible to realize a riskless profit using an arbitrage portfolio.

Example 1: $\mathbf{DP > F}$

- Consider a forward contract with an underlying investment asset. The spot price of this asset at the inception of the contract is equal to \$40. The time to maturity τ of the forward contract is equal to three months. The interest rate is equal to 5% p.a. c.c. The contracted delivery price is equal to \$43 .

- The forward price is given by:

$$\begin{aligned} F &= S_{t_0} \cdot e^{r\tau} \\ &= 40 \cdot e^{0.05 \cdot 0.25} \\ &= 40.50 \end{aligned}$$

- At t_0 we undertake the following actions:
  1. We borrow \$40 at %5 for a period of three months. This involves a positive cashflow of \$40.
  2. We buy one unit of the underlying asset spot at t_0. This involves a negative cashflow of magnitude \$40.
  3. We enter in a short forward contract with a time to maturity of three months.
- At maturity T we undertake the following actions:
  1. We sell the underlying asset spot. This involves a positive cashflow of magnitude \$43.
  2. We repay the borrowed amount plus interest equaling. This involves a negative cashflow of magnitude \$40.50.
- We realized a riskless profit equaling to:

$$\$43 - \$40.50 = \$2.50$$

Example 2: DP < F

- Consider a forward contract with an underlying investment asset. The spot price of this asset at the inception of the contract is equal to \$40. The time to maturity τ of the forward contract is equal to three months. The interest rate is equal to 5% p.a. c.c. The contracted delivery price is equal to \$39 .
- The forward price is given by:

$$\begin{aligned} F &= S_{t_0} \cdot e^{r\tau} \\ &= 40 \cdot e^{0.05 \cdot 0.25} \\ &= 40.50 \end{aligned}$$

- At t_0 we undertake the following actions:
  1. We short one unit of the underlying asset. We realize a positive cashflow of magnitude \$40.
  2. We invest the proceeds of our short operation at the interest rate of 5%.
  3. We enter into a long forward contract to buy the underlying asset in three months. The delivery price is equal to \$39.

- At maturity T we undertake the following actions:
  1. We buy the asset via the forward contract at the contracted delivery price. This involves a negative cashflow of magnitude \$39.
  2. We close out the short position with the assets we bought via the forward contract.
  3. We receive the proceeds of the investment. This involves a positive cashflow of magnitude \$40.50.
- We realized a riskless profit equaling to:

$$\$40.50 - \$39. = \$1.50$$

Conclusion

Both types of arbitrage will force the delivery price back to the forward price. This is the reason why the forward price is also called the arbitrage-free forward price. The first type of arbitrage operation is called a cash-and-carry arbitrage operation; the second type of arbitrage operation is called a reverse-cash-and-carry arbitrage operation.

3.3.11 Replicating portfolio's when the underlying is a dividend-paying asset

- It is possible that the asset that underlies a forward contract has holding costs and/or holding benefits. Examples of holding benefits are dividends and coupons. Examples of holding costs are storage costs and insurance costs.
- When constructing a replication portfolio, one needs to take into account these costs and benefits. This means that the replicating portfolio also has to replicate these additional cashflows.

The discrete dividend case

- Consider an investment asset underlying a forward contract provides a single discrete dividend during the lifetime of the forward contract. Suppose the dividend is certain and the magnitude is know and fixed.
- When a stock pays out a dividend, the price of that stock will fall with an amount equal to that dividend at the time the stock starts trading ex-dividend. Suppose S_{t_1} is the moment just before the stock starts to trade ex-dividend or the last moment the stock traded cum-dividend. When the stock starts trading ex-dividend at time t_2, the stock price will fall to:

$$S_{t_2} = S_{t_1} - D$$

- We can express the price of the stock at time T as follows:

$$S_T = S_{t_1} - D \cdot e^{r \cdot (T-t)} + \Delta S$$

  The dividend is considered as a negative cashflow because the value of the underlying is lowered by the magnitude of the dividend at ex-dividend day. A dividend can thus be considered as a holding benefit for the short party. The forward price is calculated by compound the spot price of the underlying S_{t_0}:

$$F = S_{t_0} \cdot e^{r\tau}$$

  Because the value of the underlying will decrease during the lifetime of the forward; the forward price as calculated above would be too high. Therefore, the forward price needs to take into account that the value of the underlying will decrease during the lifetime of the forward contract.

- We can write the payoff of the long forward contract as:

$$\begin{aligned} f_T^{long} &= S_T - DP \\ &= S_{t_0} + \Delta S - D \cdot e^{r \cdot (T-t_2)} - DP \\ &= S_T - D \cdot e^{r \cdot (T-t_2)} - DP \end{aligned}$$

- The goal is to construct a replicating portfolio for this payoff function.
- At t_0 we need to undertake the following actions:
  - Buy the underlying at the current spot price. This involves a negative cashflow with a magnitude of S_{t_0}.
  - Take out a loan for the present value of the dividend plus the present value of the delivery price. This involves a positive cashflow of magnitude $PV(D) + PV(DP)$.
- At maturity day T we need to undertake the following actions:
  - We sell the underlying at the current spot price. This involves a positive cashflow of magnitude S_T.
  - We pay back the loan. This involves a negative cashflow of magnitude $D \cdot e^{r \cdot \tau} + DP$.
- The cost of the replicating portfolio is equal to the sum of all cashflows that take place at t_0:

$$-S_{t_0} + D \cdot e^{-r \cdot (t_2 - t_0)} + DP \cdot e^{-r \cdot (T - t_0)}$$

- The payoff of the replicating portfolio at maturity is equal the sum of all cashflows that take place at maturity T::

$$S_T - D \cdot e^{r \cdot \tau} - DP$$

- We can easily see that the payoff of the replicating portfolio is equal to the payoff of the forward contract we considered earlier. Therefore, the cost of the replicating portfolio has to be equal to the cost of the forward contract:

$$P^{\,long\ fw.} = P^{\,repl.\ port.} = -S_{t_0} + D \cdot e^{-r \cdot (t_2 - t_0)} + DP \cdot e^{-r \cdot (T - t_0)}$$

- We know the value of a forward contract is zero at it's inception. Therefore, the following statement has to hold:

$$\begin{aligned} 0 &= -S_{t_0} + D \cdot e^{-r \cdot (t_2 - t_0)} + DP \cdot e^{-r \cdot (T - t_0)} \\ DP &= \left(S_{t_0} - D \cdot e^{-r \cdot (t_2 - t_0)}\right) \cdot e^{r \cdot (T - t_0)} \end{aligned}$$

Example

- Consider a zero-cost forward contract with a time to maturity of nine months, where the spot price of the underlying asset at the time of the inception of the contract is equal to \$100. The interest rate is 5% p.a. c.c. A dividend of 5% on the underlying asset is expected within five months.
- We are interested in the forward price of this forward contract which can be calculated in the following manner:

$$\begin{aligned} DP &= (S_{t_0} - D \cdot e^{-r\cdot(t_2-t_0)}) \cdot e^{r\cdot(T-t_0)} \\ &= (100 - 5 \cdot e^{\frac{5}{12}\cdot 0.05}) \cdot e^{0.05\cdot(\frac{9}{12})} \\ &= \$98.737 \end{aligned}$$

The multiple dividend case

- We are interested in determining the forward price of an asset with multiple dividends.
- The basic philosophy stays the same. We discount the incoming cashflows and subtract them from the spot price of the underlying asset. This amount is then compounded up to maturity.
- Suppose we did not subtract the discounted incoming cashflows from the spot price at the inception of the contract. In that case, the delivery price would be higher. We would compound the spot price of the underlying at the inception of the contract up to maturity. However, the underlying asset loses some of its value throughout its lifetime. The compounded spot price of the underlying at the time of the inception of the contract does not take this into account. The delivery price is therefore too high because it does not accurately reflect the price of the underlying at maturity.
- In the case of a stock, the value of that is stock is lowered when the stock starts to trade ex-dividend with an amount equal to the value of the dividend.
- In the case of a bond, the value of that bond is lowered when a coupon is paid out, with an amount equal to the value of that coupon.
- We can calculate the forward price, using the following equation:

$$F = \left[S_{t_0} - \sum_i D_i^{(t_i - t_0)}\right] \cdot e^{r\cdot\tau}$$

Example

- Consider a 9-month forward contract on a coupon-paying bond. The bond price at the inception of the forward contract is equal to \$95. Coupons are paid every three months. The coupon rate is equal to 5%. the term structure of interest rates is flat at 10% p.a. c.c. The face value of the bond is equal to \$100.
- We are interested in the forward price of this forward contract.

- We can calculate the forward price in a similar way to the previous examples. In this case we have to discount multiple cashflows. The value of the bond will drop each time a coupon is payed out. In the most simple case, the forward price is equal to the compounded value of the spot price of the underlying at the inception of the contract. This forward price would be too high because the value of the coupon-paying bond decreases throughout it's lifetime by paying out coupons. The forward price has to take this into account. This is done by discounting all the coupons back to the time of the inception of the contract. The discounted coupons are then summed and subtracted from the spot price of the bond at that time. This difference is then compounded up to the maturity of the forward contract.

  This compounded difference is equal to the forward price:

$$\begin{aligned} F &= \left[S_{t_0} - \sum_i D_i^{(t_i - t_0)}\right] \cdot e^{r \cdot \tau} \\ &= \left[95 - 5 \cdot e^{-0.25 \cdot 0.10} - 5 \cdot e^{-0.5 \cdot 0.10}\right] \cdot e^{0.10 \cdot \frac{9}{12}} \\ &= \$92.02 \end{aligned}$$

- Suppose the quoted delivery price did not take into account the different dividends. The delivery price would clearly be too high:

$$\begin{aligned} DP &= S_{t_0} \cdot e^{r \cdot \tau} \\ &= 95 \cdot e^{0.10 \cdot \frac{9}{12}} \\ &= 102.40 \end{aligned}$$

- In that case we can easily set up an arbitrage portfolio:
  - At time t_0:
    1. We buy the underlying spot. This involves a negative cashflow of magnitude \$95.
    2. We enter in a short forward position to sell the underlying at maturity T.
    3. We enter a loan for the present value of the first dividend. This involves a positive cashflow of magnitude $\$5 \cdot e^{-0.25 \cdot 0.10} = \$4.88$.
    4. We enter a loan for the present value of the second dividend. This involves a positive cashflow of magnitude $\$5 \cdot e^{-0.5 \cdot 0.10} = \$4.76$.
    5. We enter a loan with a nominal value equal to the spot price of the underlying asset minus the nominal values of the loan of the first and the second dividend. This involves a positive cashflow of magnitude $\$95 - \$4.88 - \$4.76 = \$85.36$. We can therefore finance the underlying with the total amount of loans we entered.

- At time t_1:
  1. We receive our first dividend. This involves a positive cashflow of magnitude \$5.
  2. We pay back the loan with nominal value of the first dividend plus interest. This involves a negative cashflow of magnitude \$5.
- At time t_2:
  1. We receive our second dividend. This involves a positive cashflow of magnitude \$5.
  2. We pay back the loan with nominal value of the second dividend plus interest. This involves a negative cashflow of magnitude \$5.
- At maturity T:
  1. We sell the underlying asset via the forward contract. This involves a positive cashflow of magnitude \$102.40.
  2. We pay back the loan with a nominal value equal to the spot price of the underlying at t_0 minus the nominal values of the loan of the first and the second dividend plus interest. This involves a negative cashflow of magnitude $85.36 \cdot e^{0.05 \cdot \frac{9}{12}} = \$88.62$.
  3. We realized a riskless profit equal to $\$102.40 - \$88.62 = \$13.78$

The continuous dividend case

- We are interested in determining the forward price of an asset that provides income in a continuous manner.
- The basic philosophy stays the same. We discount the incoming cashflows and subtract them from the spot price of the underlying asset. This amount is then compounded up to maturity.
- In this case, we discount the spot price of the underlying at the time of the inception of the contract using a dividend yield δ.
- The forward price of an asset that provides income in a continuous manner with dividend yield δ is equal to:

$$\begin{aligned} F &= S_{t_0} \cdot e^{-\delta\tau} \cdot e^{r\tau} \\ &= S_{t_0} \cdot e^{(r-\delta)\cdot\tau} \end{aligned}$$

Example

- Currencies are an example of assets that provide income in a continuous manner.
- Suppose the USD trades spot at €S_{t_0}. The interest rate in the EUR is equal to r% p.a. c.c. The interest rate on the USD is equal to δ% p.a. c.c.
- We are interested in the forward price of a τ-year, zero-cost forward contract where the underlying is one USD.
- The USD provides continuous income in the form of an interest rate δ. When calculating the forward price of one USD, the spot price of one USD at the time of the inception of the forward contract needs to be reduced with the present value of all future interest payments. We determine the forward price by compounding this amount up to maturity. Because the spot price is expressed in EUR, we need to use the EUR interest rate. Mathematically, this becomes:

$$F = S_{t_0} \cdot e^{(r-\delta)\cdot\tau}$$

Forward contracts on stock indices

- It is not practical to model and track every dividend of every stock individually. Therefore, one can interpret the stock index as a continuous dividend paying asset. In that case, the forward price is given by:

$$F = S_{t_0} \cdot e^{r_{cc}-\delta_{cc}}$$

- We compute δ as the average dividend yield per annum during the lifetime of the forward contract. Mathematically this becomes:

$$\delta = \frac{\gamma_{d_1} + \gamma_{d_2} + ... + \gamma_{d_n}}{n}$$

Summary

- If the underlying provides income in a discrete manner, the forward price is given by:

$$
\begin{aligned}
F &= \left[S_{t_0} - \sum_i D_i^{(t_i - t_0)}\right] \cdot e^{r \cdot \tau} \\
&= S_{t_0} \cdot (\frac{1+r}{1+\delta})^{\tau}
\end{aligned}
$$

- If the underlying provides income in a continuous manner, the forward price is given by:

$$F = S_{t_0} \cdot e^{r-\delta}$$

- We have made some implicit assumptions throughout this chapter, being:
  - There are no transaction costs.
  - There are no restrictions on short selling.
  - The borrowing and the lending interest rates are the same.

  If the assumptions are relaxed, we get an arbitrage-free band of delivery prices instead of a single arbitrage-free delivery price.

3.4 Determining the marked-to-market value of a position in a forward contract

- To determine the value of the forward contract, at maturity T, we compare two cashflows:
  - The delivery price of the forward contract DP. This is the contracted price at which the underlying changes hands.
  - The spot price of the underlying at maturity S_T.
- Now, we want to determine the value of a forward contract at any point in time. The same logic still applies id est the value of a position in a forward contract is still a comparison between the spot price of the underlying at that point in time and the forward price. However, we need to take into account that we are now comparing cashflows at different points in time. Therefore, we need to discount the forward price to the point in time of comparison. For example, the value of a long position in a forward contract, at time t_0 is given by:

$$\begin{aligned} f_{t_0} &= S_{t_0} - (S_{t_0} \cdot e^{r\tau}) \cdot e^{-r\tau} \\ &= S_{t_0} - S_{t_0} \\ &= 0 \end{aligned}$$

- The value of a position in a forward contract is thus calculated by comparing buying spot and buying forward and aligning the cashflows that result from these operations, in time.
- Remarks:
  - A forward contract has a value of zero, when the delivery price is set to the arbitrage-free delivery price. This is the case at the inception of the forward contract, when the forward contract is negotiated. Mathematically:

$$f_{t_0} = 0$$

  - During the lifetime of the forward contract, a position may gain or lose value.
  - At maturity, the value of a position in the forward contract is equal to the payoff of that position:

$$f_T^{long} = S_T - DP$$

  - At maturity, the value of a position in the forward contract, is determined by comparing the acquisition cost spot and the acquisition cost contracted forward.

- We look at the following example:
  - Consider a forward contract with a time to maturity of six months. The underlying is a zero-coupon bond that will mature in one year from now. The nominal value of the bond is equal to \$950. The price of the bond at this time is equal to \$930. The term structure of interest rates is flat at 6% p.a. c.c.
  - We want to determine the actual value of a long position in this forward contract. There are two ways we can calculate the value of this position:
    1. We can compare entering in a new forward contract with the old forward contract. The forward price for the new forward contract is given by:

$$\begin{aligned} F_{new} &= S_{t_0} \cdot e^{r\tau} \\ &= 930 \cdot e^{0.5 \cdot 0.06} \\ &= 958.32 \end{aligned}$$

       We compare entering into a new forward contract with the old forward contract:

$$F_{new} - F_{old} = 958.32 - 950 = \$8.32$$

       The forward price of the new contract is higher than the forward price of the old contract. Therefore, the old forward contract has a positive value for the long position and a negative value for the short position in the contract. The value of the old forward contract is ±\$8.32 at maturity.
    2. We can compare buying the underlying asset spot with the present value of buying the underlying via the forward contract:

$$\begin{aligned} S_{t_1} - PV(F) &= 930 - 950 \cdot e^{-0.5 \cdot 0.06} \\ &= 930 - 921.92 \\ &= \$8.08 \end{aligned}$$

       The value of the old forward contract is ±\$8.08 at the present point in time.

3.5 Determining the forward price for consumption assets

3.5.1 No-arbitrage

Pure consumption assets do not provide income and need to be stored. In some cases the no-arbitrage principle breaks down. In that case, it is impossible to determine a forward price.

1. Suppose $F > (S + u) \cdot e^{r\tau}$. In this case, the delivery price is lower than the arbitrage-free forward price. The question ad hand is if it is possible to take advantage of this low delivery price. We can theoretically set up an arbitrage portfolio by:

    - Taking out a loan for the amount of the spot price of the underlying asset at t_0 and the present value at t_0 of the storage costs. The interest rate on the loan is $r\%$. This entails a positive cashflow of magnitude $S_{t_0+U_{t_0}}$.
    - Buying the underlying spot at t_0. This entails a negative cashflow of magnitude S_{t_0}.
    - Storing the underlying asset up to maturity at cost U_T.
    - Going short in a forward contract.

    At maturity, the following steps unfold:

    - We sell the underlying via the forward contract. This entails a positive cashflow of magnitude F.
    - We repay the loan for the amount of the spot price of the underlying asset at t_0 and the present value at t_0 of the storage costs plus interests. This entails a negative cashflow of magnitude $(S_{t_0} + U_{t_0}) \cdot e^{r\tau}$.

    We realize a riskless profit of magnitude:

    $$F - (S_{t_0} + U_{t_0}) \cdot e^{r\tau}$$

    There were no obstacles in setting up this arbitrage portfolio. Therefore, in this case, arbitrage is possible.

2. Suppose $F < (S + u) \cdot e^{r\tau}$. In this case, the delivery price is higher than the arbitrage-free forward price. The question ad hand is if it is possible to take advantage of this high delivery price. We can theoretically set up an arbitrage portfolio by:

    - Going long in a futures contract.
    - Going short in the underlying asset.

    In this case, we immediately encounter a problem. This is because the underlying asset has alternative uses. Therefore, owners might be reluctant to enter short selling constructions. To do this kind of arbitrage, you need to get your hands on the commodity itself. If nobody wants to lend the commodity, arbitrage is not possible. In general, arbitrage works if the commodity is abundantly available. However, the scarcer the commodity, the less arbitrage will be possible.

From all of the above we must conclude that we cannot determine a forward price for some pure consumption assets. The only relationship that still holds is given by:

$$F \leq (S_{t_0} + U_{t_0}) \cdot e^{r\tau}$$

Using this equation we can calculate the tension in the market of a given commodity. The bigger the inequality, the bigger the tension.

3.5.2 The cost of carry and the convenience yield

3.5.3 Conclusion

We can conclude that the no-arbitrage principle to determine the forward price of a certain asset does not work when:

- The asset is a non-tradable asset.
- The asset is a non-storable asset.

In these cases, we can only look at demand and supply to come up with a forward price. It is not possible to determine a theoretical forward price.

4 Future contracts

4.1 Futures pricing

Futures contract are similar to forward contracts. There are however two big differences

- Future contracts have delivery options.
- Futures contracts have a marking-to-market procedure. This means futures contracts are settled daily.

The question at hand is whether the delivery options that come with future contract and the daily settlement have an impact on pricing.

4.1.1 Delivery options

- The party that is short in the futures contract will have the delivery options. These Delivery options are a benefit from the point of view of the party that is short because they grant some leeway in the delivery of the underlying assets.
- The short party finds future contracts more interesting than forward contracts. All else being equal, there will be more sellers and less buyers in the future market than in the forward market. Therefore, the future price will be lower than the forward price. The futures price will thus be lower than the forward price because of the presence of delivery options.
- The party that is long in the futures contract will have to pay a lower price in comparison with the forward price. This is because he is uncertain about the quality of the underlying that is going to be delivered.
- Delivery options make future contract less suitable for a long hedge. This is because the hedge will be less precise due to the presence of delivery options.
- Exchanges try to limit the amount of delivery options in future contracts. This in turn limits the value of these delivery options. Therefore, the discrepancy between the forward and the future price is in practice rather small.
- On average the economical impact of choosing between different types of underlying is quite small. In general, delivery options do not play an important role.

4.1.2 The marking-to-market procedure

- A theoretical proof exists (Cox, Ingersoll and Ross)that forward and future prices are equal when:
  - Interest rates are constant.
  - Interest rates are a deterministic function of time. This condition can be relaxed somewhat. In practice it is sufficient to assume that interest rates are forecastable.
- Consider a futures contract with a time to maturity of τ. The future's price at the end of day t is denoted by F_t. The forward price, at the end of day t is denoted by G_t. The risk-free rate on a daily basis, is denoted by δ.
- We are interested in comparing two strategies. The first strategy makes use of the future contract. The second strategy makes use of the forward contract. Our goal is to show that the payoff's of both strategies, calculated at maturity are equal to one another. If this is the case, the forward and future prices have to be equal because one of the portfolios is based on future contract and the other one on forward contracts.
- We first consider the future strategy.
  - The future strategy comprises the following steps:
    1. Investing the amount equal to the futures price F_{t_0} in a riskless bond at time t_0.
    2. Entering in a long future position with an underlying value of e^{δ} at time t_0.
    3. Increasing the underlying value of the long future position to $e^{i\delta}$ at the end of day t_i.
  - The results of this strategy are presented in the table below:

Day	0	1	2	...	T
Futures price	F_{t_0}	F_{t_1}	F_{t_2}	...	F_T
Position	e^{δ}	$e^{2\delta}$	$e^{3\delta}$	...	$e^{T\delta}$
Gains/losses	0	$(F_{t_1} - F_{t_0})e^{\delta}$	$(F_{t_2} - F_{t_1})e^{2\delta}$	...	$(F_T - F_{T-1})e^{T\delta}$
Result at T	0	$(F_{t_1} - F_{t_0})e^{T\delta}$	$(F_{t_2} - F_{t_1})e^{T\delta}$	...	$(F_T - F_{T-1})e^{T\delta}$

– We can calculate the value of the long future position at maturity T. This is done by compounding all interim cashflows up to maturity and taking the sum of these compounded cashflows. Suppose $F_{t_i} - F_{t_{(i-1)}}$ represents the contribution of each day i to the total value of the position at maturity T. The value of the long future position at maturity T is then given by:

$$\begin{aligned} f_T^{long\ fut} &= \sum_{i=1}^{T}(F_{t_i} - F_{t_{(i-1)}}) \cdot e^{T\delta} \\ &= (F_T - F_{t_0}) \cdot e^{T\delta} \\ &= (S_T - F) \cdot e^{T\delta} \end{aligned}$$

– We can also calculate the value of the bond at maturity T. This is nothing else then the initial investment in the bond F compounded up to maturity T. Mathematically, the value of the bond at maturity is given by:

$$f_T^{bond} = F \cdot e^{\tau\delta}$$

– The value of the entire strategy consists of the value of the long position at maturity T plus the value of the bond at maturity T. The value of the entire strategy, at maturity T, is therefore given by:

$$\begin{aligned} f_T &= F \cdot e^{T\delta} + (S_T - F) \cdot e^{T\delta} \\ &= S_T \cdot e^{T\delta} \end{aligned}$$

– In summary, the initial investment in this strategy is equal to F_{t_0}. The payoff at maturity of this strategy is equal to $S_T \cdot e^{T\delta}$.

- Next, we consider the forward strategy.
  – The forward strategy comprises the following steps:
    1. Investing the amount equal to the forward price G_{t_0} in a riskless bond at time t_0.
    2. Entering in a long forward position with an underlying value of e^{δ} at time t_0.

  – The formula for the payoff of a long forward contract at maturity T is given by:

$$f_T^{long\ fw} = S_T - G_{t_0}$$

  – We can also calculate the value of the bond at maturity T. This is nothing else then the initial investment in the bond G compounded up to maturity T. Mathematically, the value of the bond at maturity is given by:

$$f_T^{bond} = G \cdot e^{\tau\delta}$$

– The value of the entire strategy consists of the value of the long position at maturity T plus the value of the bond at maturity T. The value of the entire strategy, at maturity T, is therefore given by:

$$
\begin{aligned}
f_T &= G \cdot e^{T\delta} + (S_T - G) \cdot e^{T\delta} \\
&= S_T \cdot e^{T\delta}
\end{aligned}
$$

– In summary, the initial investment in this strategy is equal to G_{t_0}. The payoff at maturity of this strategy is equal to $S_T \cdot e^{T\delta}$.

- We have shown that both investment strategies yield the same terminal cashflow $S_T \cdot e^{T\delta}$. The investment of the first strategy amounts to F_{T_0} while the invested amount of the second strategy is equal to G_{t_0}. Since the terminal cashflows are the same, the initial invested amount also needs to be the same. This means that the future price needs to be equal to the forward price. Mathematically, this becomes:

$$F_{t_0} = G_{t_0}$$

- We showed that, when interest rates are constant, the forward price and the future price are equal to one another.
- What if the CIR-assumption does not hold? Id est what happens when interest rates move in an unpredictable manner?

– The only difference between a future contract and a forward contract is the marking-to-market process.

– Suppose the underlying asset is positively correlated with interest rates. When the interest rates go up, the asset price also goes up. The party that is long in the future contract will gain when this is the case. However, because of the higher interest rates, the long can now also invest his profits at a higher yield. Conversely, when interest rates decline, the price of the underlying asset will also decline. The party that is long in the futures contract loses in this case. However, because of the lower interest rates, he can fund his losses at a lower interest rate. In summary, the long is better of in all possible situations. Therefore, the futures price will be higher than the forward price in the case where interest rates are positively correlated with the price of the asset underlying the contract.

– Suppose the underlying asset is negatively correlated with interest rates. When the interest rates go up, the asset price goes down. The party that is long in the future contract will lose when this is the case. Because of the higher interest rates, the long now has to fund his losses at a higher interest rate. Conversely, when interest rates decline, the price of the underlying asset will rise. The party that is long in the futures contract gains in this case. However, because of the lower interest rates, he must invest his profits at a lower yield. In summary, the long is worse of in all possible situations. Therefore, the futures price will be lower than the forward price in the case where interest rates are negatively correlated with the price of the asset underlying the contract.

– In summary, the following statements hold:

$$\rho(P, r) > 0 \rightarrow F_{Fut.} > F_{For.}$$
$$\rho(P, r) < 0 \rightarrow F_{Fut.} < F_{For.}$$

5 Hedging with forward and future contracts

5.1 Introduction

- Hedging is defined as the reduction of uncertainty in future cashflows. If no uncertainty is left after hedging an exposure to a certain risk, the hedge is called a perfect hedge.
- The most important economic function of forward and future contracts is actually hedging.
- In the first place, we are are interested in possible reasons for hedging. Id est why would a firm want to hedge an exposure to a certain risk.
  - An important question to consider is whether a firm should hedge risks away that the shareholder can hedge away himself? If the firm hedges some risks, it will generate costs in doing so. This will almost certainly not be to the liking of some investors.
  - Consider the following example. Nestle is a company based in Switzerland but has many investors settled in the United States. Nestle could choose to hedge it's Dollar exposure. For an investor based in the USA however, the Dollar exposure of Nestle is not a problem. On the contrary, by hedging away it's Dollar exposure, Nestle would create a risk for the investor based in the USA. This is because he uses the USD as his currency of choice.
  - The most compelling argument for hedging is that hedging reduces the volatility of future cashflows. This can be important for tax-purposes. This is because in many countries, tax rates are progressive. By reducing the volatility of cashflows, the firm ensures its profits do not end up in the higher tax brackets.

5.2 The long hedge and the short hedge

We assume a positive correlation between the price of the asset underlying the forward contract and the forward price. In that case, the two different types of hedges and their use cases are:

1. The long hedge. This hedge is used when we want to buy an asset in the future. The hedge allows us to reduce the uncertainty in the price of the asset. To do this, we need to take the following steps:
   - Take a long position in a forward or future contract at t_0.
   - Buy the asset that is being hedged spot at the maturity of the hedge T.
   - Close out the forward or future contract at maturity T.
2. The short hedge. This hedge is used when we want to sell an asset in the future. The hedge allows us to reduce the uncertainty in the price of the asset. To do this, we need to take the following steps:
   - Take a short position in a forward or future contract at t_0.
   - Sell the asset that is being hedged spot at the maturity of the hedge T.
   - Close out the forward or future contract at maturity T.

5.3 Imperfect hedges

When hedging, it is possible that the forward price, for the original forward contract, of the underlying of the forward contract at the maturity of the forward contract will not be equal to the spot price at the maturity of the hedge of the asset that is being hedged. Id est $F_T \neq S_T$. Tere are two possible reasons for this:

1. A mismatch between the term of the forward contract and the time over which one wants to hedge. Id est the maturity of the forward contract is not equal to the maturity of the hedge.
2. A commodity mismatch. The asset underlying the forward contract is not exactly the same as the asset that is being hedged. Therefore differences are possible between the forward price at maturity T of the original forward contract and the spot price of the asset that is being hedged at time T.

In the case of future contracts there is almost certainly a mismatch for the delivery date. This is because futures usually have a delivery or maturity month. When the a party enters the delivery month, he has to take delivery or make delivery. Therefore, parties usually close out their positions before the maturity month of the futures contract. In doing so, they do not have to make or take delivery.

We conclude that the property $F_{T,T} = S_T$ does not hold up when hedging. In general, the hedge will be unwound before maturity and the asset underlying the forward contract is not exactly the same as the asset that is being hedged.

Consider a long position in a futures contract that matures at maturity T. The hedge is unwound at time t, which is before maturity T. At the same time t we buy the underlying spot in the market. The price we end up paying for the underlying is calculated by taking the sum of all cashflows:

$$\begin{aligned} P &= -S_t + (F_{t,T} - F_{t_0,T}) \\ &= -F_{t_0,T} - S_t + F_{t,T} \\ &= -(F_{t_0,T} + S_t - F_{t,T}) \\ &= -(F_{t_0,T} + b_t) \end{aligned}$$

We define the difference between the spot price and the futures price at time t as the basis b_t at that time. Mathematically, this becomes:

$$b_t = S_t - F_t$$

The term basis risk refers to the risk that remains after hedging because the convergence of the future price towards the spot price at maturity does not take place. Id est there is some risk left because of the difference between the future price and the spot price at maturity. The basis risk will increase if the basis strengthens. The basis risk will decrease if the basis declines.

5.4 Optimizing the hedge

The goal is of course to create a perfect hedge. We can do this by choosing an underlying asset that closely resembles that asset we want to hedge and by minimizing the maturity mismatch.

The maturity mismatch originates from the fact that we want to unwind the hedge prematurely. The future contract ends at maturity T but is unwound at time t. To minimize the maturity mismatch, we choose the delivery month that is as close as possible to, but later than, the end of the period over which we wish to hedge.

Furthermore, future prices during the month of the expiration of the future contract can be erratic. This is just another reason to choose a forward contract with a longer lifespan than the period over which we wish to hedge.

5.4.1 Example

- Consider the following example:
  - A US-corporate will receive ¥50.000.000 at the end of July. There are future contracts available for the JPY with different maturities. All future contract expire at the end of the month. The available maturities are March, June, September and December. The contract size of all future contracts is ¥12.500.000. The goal is to hedge this position using future contracts.
  - In general, we need to close out the future position, at the end of the period over which we wanted to hedge. To minimize the maturity mismatch we should choose the delivery month as close as possible to, but lather than the end of the period over which we want to hedge. The end of July marks the end of the hedging period. September is the closest month to July that is later than July, for which a future contract is available. Therefore, we choose to short four September future contracts.
  - Suppose the future price on March first is equal to 0.7800 c/¥:

$$F_{t_0,T} = 0.7800 \ c/¥$$

  - At the end of July the future price is equal to 0.7250 c/¥:

$$F_{t,T} = 0.7250 \ c/¥$$

  - At this time we need to close out the position. We therefore realize a profit on the future position equal to:

$$\begin{aligned} f_t^{short\ fut.} &= F_{t_0,T} - F_{t,T} \\ &= 0.7800 - 0.7250 \\ &= ¥0.055 \end{aligned}$$

  - At the same time the company will receive the amount in JPY. The company should sell these JPY spot. The spot price at that time is equal to 0.7200 c/¥.
  - The goal of our hedge was to lock in the price of the JPY. With our hedge, we ended up receiving the following price:

$$\begin{aligned} P &= F_{t_0,T} + b_t \\ &= 0.7200 + 0.055 \\ &= 0.7750 \end{aligned}$$

  - Remark that in theory we cannot compute the difference between $F_{t_0,T}$ and $F_{t,T}$. This is because both forward prices reflect cashflows at different point in time. What we should do is compound the forward price of our forward contract at time t_0 to time t. However, the time gap is usually small and is therefore often ignored.

5.4.2 Rolling the hedge

It is possible a firm needs to hedge an exposure over a period that is longer than the time to maturity of the available forward contracts. If this is the case, it can role over the hedge. This is done by dividing the lifespan of the hedge into different, smaller periods. The firm then sets up a hedge at the start of each of these periods and closes the hedge before then end of this period to avoid the maturity month.

Example

- A company wants to set up a hedge in April of year x_0 to sell 100.000 barrels of oil in June of year x_1. The spot price at time t_0 is equal to $\$19/barrel$. Future contracts are available up to six months. The contract size of a future contract is equal to 100 barrels. Suppose the spot price for oil in June of the next year is equal to $\$16/barrel$.
- To set up a hedge, the firm takes the following steps:
  - The firm goes short in 100 October future contracts.
  - It rolls over the October future contracts to March futures, in September.
  - It rolls over the March future contracts to July futures, in February.
- Rolling over the hedge consist of closing out the existing future position and entering a new future position at the same time. Each position is closed out in the month before the delivery month.
- We now want to calculate the total profit or loss the firm incurred over the lifespan of the hedge. For each future contract, we need can calculate the payoff using the following formula:

$$f_t = F_t - F_{t_0}$$

- The future prices are given by:

Month	Year	F_{t_0}	F_t	$f_t = F_t - F_{t_0}$
October	x	\$18.20	\$17.40	\$0.80
March	$x+1$	\$17.00	\$16.50	\$0.50
July	$x+1$	\$16.30	\$15.90	\$0.40
Total				\$1.70

- The forward contracts provide a positive cashflow of \$1.70 per contract. However, the spot price evolved from $S_{t_0} = \$19$ to $S_t = \$17$. This means the company made a loss by postponing their purchase. This is the loss the firm would have incurred when it would have not hedged it's exposure. The loss amounts to:

$$\begin{aligned} -f_T &= S_{t_0} - S_t \\ &= \$19 - \$17 \\ &= \$2 \end{aligned}$$

- In total, the firm still incurs a loss. However, because the firm hedged it's position, the loss is smaller than it would have been without setting up a hedge.
- The loss is due to closing out the future positions before the maturity of the future contract. The future price and the spot price converge at the maturity of the contract. However, the firm has to close out is's position before the maturity month. In that case, the future price $f_{t,T}$ is not equal to the spot price S_t. The payoff of the future contract at time t is equal to:

$$f_t^{short\ fut.} = F_{t,T} - F_{t_0,T}$$

  This is also the case when the future contract is not being rolled over. However, when rolling a hedge, the future position is closed out multiple times. Every time the future position is closed out, there will exist a difference between the future price and the spot price. In conclusion, the difference in the spot price over the period that is being hedged will not be equal to the sum of the differences between the forward price at time t_{i+1} and the forward price at time t_i:

$$S_T - S_{t_0} \neq \sum_{i=0}^{nper} F_{t_{i+1}} - F_{t_i}$$

The case without a maturity mismatch

Consider the case where:

- A future contract was available that could span the period between T and t.
- The future contract has the same maturity as the hedge, namely T.
- It is possible to close out the future contract at the maturity T of the contract.

In that case:

- The forward price at the maturity would be equal to the spot price of the underlying at that time:

$$S_T = F_T$$

- The losses incurred by delaying the sale would be fully compensated by the proceeds of the position in the future contracts:

$$S_T - S_{t_0} = F_T - S_{t_0}$$

Metallgesellschaft AG

Consider the case of Metallgesellschaft AG. The company set up a rolling hedge with short term future contracts. However, these future contracts have a margin requirement. As a consequence, there were margin calls on a regular basis. This put pressure on the cashflow of the company. The board of directors did not understand the workings of hedging and therefore decided to close out the position. In doing so, the company incurred a loss of $1.33 billion. This is a good example of timing problems that can arise by hedging an exposure.

5.4.3 Commodity mismatches

- A commodity mismatch is caused by one of two reasons:
    1. Delivery options with respect to the grade of the underlying commodity.
    2. Cross-hedging. This is a hedge for one commodity which is accomplished with future contracts on another product.
- Cross-hedging is done when there are no futures on the asset that is being hedged. The hedgers chooses a contract for which the changes in the future price are highly correlated with the changes in the price of the commodity that is to be hedged.

  We denote the future price of the contract that is used for the hedge as F^H. The theoretical future price of a contract on the actual commodity that is being hedged is denoted by F.

  The payoff at time t, in the case of a short hedge becomes:

$$\begin{aligned} f_t^{short\ fut.} &= S_t + (F_{t_0,T}^H - F_{t,T}^H) \\ &= S_t + F_{t_0,T}^H - F_{t,T}^H - S_t^H + S_t^H \\ &= F_{t_0,T}^H + (S_t^H - F_{t,T}^H) + (S_t - S_t^H) \\ &= F_{t_0,T}^H + b_t^H + (S_t - S_t^H) \end{aligned}$$

  Notice that the third term reflects how the spot price of the underlying differs from the spot price of the asset that is being hedged. The second term is the basis risk on the underlying future.
- When cross hedging we need a future contract where changes in the future price are highly correlated with changes in the price of the underlying asset. These price changes should be correlated over the hedging horizon.

5.4.4 The optimal hedge size

- We now ask ourselves what the optimal size of the hedge should be. Id est we are looking for the optimal hedge ratio h or the optimal number of contracts n times the contract size H to hedge a position of size Q. Mathematically:

$$h = \frac{n \cdot H}{Q}$$

- A naive hedge uses a hedge ratio of one. In that case, the position has been fully hedged. This means that the value of the whole position is sheltered from the risk of price changes in the underlying.
- In the presence of basis risk however, it is generally not optimal to have a hedge ratio of one. With a hedge ratio of h, the cashflows of the short hedge at time t would become:

$$f_t^{short\ hedge} = S_t + h \cdot (F_{t_0,T} - F_{t,T})$$

  We can easily rewrite these equations using the following notation:

$$\begin{aligned} \Delta S &= S_t - S_{t_0} \\ \Delta F &= F_{t,T} - F_{t_0,T} \end{aligned}$$

  The equation for the cashflows of the short hedge at time t become:

$$\begin{aligned} f_t^{short\ hedge} &= \Delta S + S_{t_0} + h \cdot (-\Delta F) \\ &= S_{t_0} + \Delta S - h\Delta F \end{aligned}$$

  Hedging is about certainty; therefore, we want to reduce the volatility of the value of the hedge. This means minimizing the variance of a change in the value of the position of the hedger:

$$\min_h var(\Delta S - h\Delta F)$$

  The variance of a change in the value of the position of the hedger is given by:

$$\begin{aligned} var(Change\ in\ Position) &= var(\Delta S - h\Delta F) \\ &= var(\Delta S) + h^2 \cdot var(\Delta F) - 2h \cdot cov(\Delta S, \Delta F) \end{aligned}$$

  Minimizing the variance of the change in the value of the position of the hedger corresponds to taking the first derivative with respect to h and equating it to zero:

$$\begin{aligned} \frac{d\ var(Change\ in\ Position)}{dh} &= 0 \\ 2h \cdot var(\Delta F) - 2 \cdot cov(\Delta S, \Delta F) &= 0 \\ \frac{cov(\Delta S, \Delta F)}{\Delta F} &= h^* \end{aligned}$$

This is indeed a minimum, since:

$$\frac{d^2\ (change\ in\ position)}{dh^2} = 2 \cdot var(\Delta F)$$
$$> 0$$

- The optimal hedge ratio is therefore equal to:

$$h^* = \frac{cos(\Delta S, \Delta F)}{var(\Delta F)}$$
$$= \rho_{\Delta S, \Delta F} \cdot \frac{\sigma_{\Delta S}}{\sigma_{\Delta F}}$$

- The correlation between ΔS and ΔF is given by:

$$\rho_{\Delta S, \Delta F} = \frac{cov(\Delta S, \Delta F)}{\sigma_{\Delta S} \cdot \sigma_{\Delta F}}$$

- We can easily see that the optimal hedge ratio depends on:
  1. The correlation between the changes in the spot price and the forward price $\rho_{\Delta S, \Delta F}$.
  2. The ratio of the two volatilities $\frac{\sigma_{\Delta S}}{\sigma_{\Delta F}}$.
- Suppose we want to hedge an exposure to a commodity that fluctuates relatively heavily. To do this, we use a forward contract that has relatively little price fluctuations. Intuitively, it is easy to see that we will need more than one forward contract to hedge the position. This is because we want to hedge large price changes with small price changes. The ratio of the two volatilities indicates how many contracts we need to hedge the exposure, per unit of risk.

5.4.5 Effect of the correlation on the optimal hedge size

We are interested in the impact of the correlation between the spot and the forward price on the optimal hedge size.

- A correlation of one means that the cashflow variance can be reduced to zero. A perfect hedge is possible.
- A correlation of zero means that whatever hedge ratio we are using, the risk will only increase, not decrease.
- A correlation between one and zero is something in between. A perfect hedge is not possible, however the exposure can be reduced.

5.4.6 Remarks

- The sign of the hedge ratio will depend on the correlation.
- If the sign is positive, the long hedge will have to take a long position in a future contract in order to offset the losses as a buyer in the spot market.
- If the sign is negative, the long hedge will have to take a short position in a future contract to offset the losses as a buyer in the spot market.
- If the correlation is zero, the hedge ratio is also zero and hedging will only increase the volatility of the position.

5.4.7 Quantifying the amount of avoided uncertainty by hedging

We may want to quantify how much uncertainty is avoided by setting up a hedge. We know that the variance in the cashflows of the position of the hedger is given by:

$$\begin{aligned} var(Change\ in\ Position) &= var(\Delta S - h\Delta F) \\ &= var(\Delta S) + h^2 \cdot var(\Delta F) - 2h \cdot cov(\Delta S, \Delta F) \\ &= var(\Delta S) + h^2 var(\Delta F) - 2h \cdot \rho_{\Delta S,\Delta F} \cdot \sigma_{\Delta S}\sigma_{\Delta F} \end{aligned}$$

The cashflow variance in case of no hedging is therefore given by:

$$var(Change\ in\ Position \mid no\ hedging) = var(\Delta S)$$

The cashflow variance in case of an optimal hedge is equal to:

$$\begin{aligned} &var(Change\ in\ Position \mid optimal\ hedge) \\ &\quad = var(\Delta S) + \rho_{\Delta S,\Delta F} \cdot \frac{\sigma_{\Delta S}}{\sigma_{\Delta F}}^2 \cdot var(\Delta F) - 2(\rho_{\Delta S,\Delta F} \cdot \frac{\sigma_{\Delta S}}{\sigma_{\Delta F}}) \cdot \rho_{\Delta S,\Delta F} \cdot \sigma_{\Delta S} \cdot \sigma_{\Delta F} \\ &\quad = var(\Delta S) \cdot (1 - \rho^2) \end{aligned}$$

We can now easily see, that by hedging the risk, the cashflow variance was reduced by:

$$\begin{aligned} &var(Change\ in\ Position \mid no\ hedging) - var(Change\ in\ Position \mid optimal\ hedge) \\ &\quad = var(\Delta S) - var(\Delta S) \cdot (1 - \rho^2) \\ &\quad = \rho^2 \cdot var(\Delta S) \end{aligned}$$

5.4.8 The effectiveness of the hedge

Recall the statistical measure R-squared R^2. R-squared R^2 is a statistical measure that represents the proportion of the variance for a dependent variable that's explained by an independent variable or variables in a regression model. Whereas correlation explains the strength of the relationship between an independent and dependent variable, R-squared explains to what extent the variance of one variable explains the variance of the second variable.

Within a univariate regression, the correlation squared ρ^2 is therefore equal to R^2:

$$\rho^2 = R^2$$

We can measure the effectiveness of a hedge by measuring the proportion of the variance that is eliminated by hedging. This is measured by R-squared R^2:

$$R^2 = \rho^2 = h^* \cdot \frac{2\sigma_{\Delta F}^2}{\sigma_{\Delta S}^2}$$

5.4.9 Example

- Consider a firm that will buy one million gallons of kerosene within three months. The firm wants to hedge this position. The standard deviation of the spot price of kerosene $\sigma_{\Delta S}$ is equal to $0,032$. The firm uses futures on Brent oil to hedge this position. The standard deviation of the future price $\sigma_{\Delta F}$ is equal to 0.040. The correlation between the spot price of the kerosene and the forward price of Brent oil forward contract $\rho_{\Delta S,\Delta F}$ is equal to 0.80. The contract size of a future contract is equal to 42.000 gallons. We want to calculate the optimal hedge ratio h^*.
- The optimal hedge ratio h^* is given by:

$$\begin{aligned} h^* &= \rho_{\Delta S,\Delta F} \cdot \frac{\sigma_{\Delta S}}{\sigma_{\Delta F}} \\ &= 0.80 \cdot \frac{0.032}{0.040} \\ &= 0.64 \end{aligned}$$

- The number of contracts needed n, to hedge the position is therefore equal to:

$$\begin{aligned} n &= h^* \cdot \frac{Q}{H} \\ &= 0.64 \cdot \frac{1.000.000}{42.000} \\ &= 15.2 \end{aligned}$$

5.4.10 Tailing the hedge

- We can take the effect of the daily settlements in a future contract into account. In doing so, we improve the optimal hedge ratio.
- Ignoring the interest that is received on the margin account, the changes in the margin account, over a hedging horizon of k days are:

$$(F_1 - F_0), (F_2 - F_1), ..., (F_k - F_{k-1})$$

- The total value of the resettlements, after K days Φ, is therefore equal to:

$$\begin{aligned}\Phi &= (F_1 - F_0) \cdot R^{k-1} + (F_2 - F_1) \cdot R^{k-2} ... + (F_{k-1} - F_{k-2}) \cdot R^1 (F_k - F_{k-1}) \cdot R^0 \\ &= \delta_{F,1} \cdot R^{k-1} + \delta_{F,2} \cdot R^{k-2} + ... + \delta_{F,k-1} \cdot R^1 + \delta_{F,k} \cdot R^0\end{aligned}$$

- We assume that the changes in the future price are identically distributed with mean δ_F and standard deviation $\sigma_{\Delta F}$. We also assume that the resettlements are independent of one another id est the future price changes are independent of one another. Therefore, the following statements hold:

$$\begin{aligned}var(x + y) &= var(x) + var(y) \\ var(x \cdot y) &= var(x) \cdot var(y) + var(x) \cdot E(y)^2 + var(y) \cdot E(x)^2\end{aligned}$$

- Using these statements, we can easily calculate the variance of the total value of the resettlements, after K days $var(\Phi)$. Mathematically, this becomes:

$$\begin{aligned}var(\Phi) &= var(\delta_{F,1} \cdot R^{k-1}) + var(\delta_{F,2} \cdot R^{k-2}) + ... + var(\delta_{F,k-1} \cdot R^1) + var(\delta_{F,k} \cdot R^0) \\ &= var(\delta_{F,1}) \cdot var(R^{k-1}) + var(\delta_{F,1}) \cdot E(R^{k-1})^2 + var(R^{k-1}) \cdot E(\delta_{F,1})^2 \\ &= \sigma^2_{\Delta F} \cdot R^{2\cdot(k-1)} + \sigma^2_{\Delta F} \cdot R^{2\cdot(k-2)} + ... + \sigma^2_{\Delta F} \cdot R^{2\cdot(1)} + \sigma^2_{\Delta F} \cdot R^{2\cdot(0)} \\ &= \sigma^2_{\Delta F} \cdot R^{2\cdot(k-1)} + \sigma^2_{\Delta F} \cdot R^{2\cdot(k-2)} + ... + \sigma^2_{\Delta F} \cdot R^2 + \sigma^2_{\Delta F}\end{aligned}$$

- For the sake of clarity, we derive the first term. All other terms are derived in the same manner.

$$\begin{aligned}var(\delta_{F,1} \cdot R^{k-1}) &= var(\delta_{F,1}) \cdot var(R^{k-1}) \\ &= var(\Delta F) \cdot var(R^{k-1}) + var(\Delta F) \cdot E(R^{k-1})^2 + var(R^{k-1}) \cdot E(\Delta F)^2 \\ &= \sigma^2_{\Delta F} \cdot 0 + \sigma_{\Delta F} \cdot R^{2\cdot(k-1)} + 0 \cdot \delta F^2 \\ &= \sigma^2_{\Delta F} \cdot R^{2\cdot(k-1)}\end{aligned}$$

- The total change in the spot price after k days is denoted by Ψ and is given by:

$$\Psi = \delta_{S,1} + \delta_{S,2} + ... + \delta_{S,k}$$

- Assume the changes in the spot price are identically distributed with variance $\sigma^2_{\delta_S}$ and mean δ_S. Furthermore, assume that these price changes are independent of one another. In that case, the variance of the total change in the spot price after k days $var(\Psi)$ is given by:

$$var(\Psi) = k \cdot \sigma^2_{\delta_S}$$

- The total cashflow that results from the hedged position is given by:

$$\Psi - h \cdot \Phi$$

- The variance of the total cashflow that results from the hedged position is given by the following equation:

$$var(\Psi - h \cdot \Phi) = \sigma^2_{\Psi} - h^2 \cdot \sigma^2_{\Phi} - 2h \cdot cov(\Psi, \Phi)$$

We now take a look at the last term, the covariance between the total change in the spot price and the total change in the forward price $cov(\Psi, \Phi)$. In general, the following properties apply for the covariance:

$$\begin{aligned} cov(X, X) &= var(X) \\ cov(X, Y) &= cov(Y, X) \\ cov(aX, Y) &= a \cdot cov(X, Y) \\ cov(X + c, Y) &= cov(X, Y) \\ cov(X + Y, Z) &= cov(X, Z) + cov(Y, Z) \end{aligned}$$

Or, more generally:

$$cov\Big(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\Big) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \cdot Cov(X_i, Y_j)$$

We can now rewrite the covariance between the total change in the spot price and the total change in the forward price $cov(\Psi, \Phi)$:

$$\begin{aligned} cov(\Psi, \Phi) &= cov(\delta_{F,1} \cdot R^{k-1} + \delta_{F,2} \cdot R^{k-2} + ... + \delta_{F,k-1} \cdot R^1 + \delta_{F,k} \cdot R^0 \text{ ; } \delta_{S,1} + \delta_{S,2} + ... + \delta_{S,k}) \\ &= cov(\delta_{F,1} \cdot R^{k-1}, \delta_{S,1}) + cov(\delta_{F,2} \cdot R^{k-2}, \delta_{S,2}) + ... + cov(\delta_{F,k}, \delta_{S,k}) \\ &= R^{k-1} \cdot cov(\delta_{F,1}, \delta_{S,1}) + R^{k-2} \cdot cov(\delta_{F,2}, \delta_{S,2}) + ... + cov(\delta_{F,k}, \delta_{S,k}) \\ &= R^{k-1} \cdot cov(\delta_F, \delta_S) + R^{k-2} \cdot cov(\delta_F, \delta_S) + ... + cov(\delta_F, \delta_S) \\ &= \Big(R^{k-1} + R^{k-2} + ... + 1\Big) \cdot cov(\delta_S, \delta_F) \end{aligned}$$

- Recall that the variance of the hedged position is given by:

$$var(\Psi - h \cdot \Phi) = \sigma^2_{\Psi} - h^2 \cdot \sigma^2_{\Phi} - 2h \cdot cov(\Psi, \Phi)$$

- To find the optimal hedge ratio, we have to minimize this expression. This is done by taking the first derivative of the expression and equating it to zero. We then rearrange the expression in terms of h. The optimal hedge ratio is denoted by h^{**} in this case. Mathematically, this becomes:

$$\frac{dvar(\Psi - h \cdot \Phi)}{dh} = \frac{d\sigma_\Psi^2 - h^2 \cdot \sigma_\Phi^2 - 2h \cdot cov(\Psi, \Phi)}{dh}$$
$$0 = 2h\sigma_\Phi^2 - 2 \cdot cov(\Psi, \Phi)$$
$$h^{**} = \frac{cov(\Psi, \Phi)}{\sigma_\Phi^2}$$

- We can now find an expression the optimal hedge ratio by substituting the equation we found for the covariance between the total change in the spot price and the total change in the forward price $cov(\Psi, \Phi)$ and the equation we found for the variance of the change in the forward price σ_Φ^2

$$\begin{aligned} h^{**} &= \frac{cov(\Psi, \Phi)}{\sigma_\Phi^2} \\ &= \frac{(R^{k-1} + R^{k-2} + ... + 1) \cdot cov(\delta_S, \delta_F)}{(R^{2\cdot(k-1)} + R^{2\cdot(k-2)} + ... + R^2 + 1) \cdot \sigma_{\Delta F}^2} \\ &= \frac{R^{k-1} + R^{k-2} + ... + 1}{R^{2\cdot(k-1)} + R^{2\cdot(k-2)} + ... + R^2 + 1} \cdot \frac{cov(\delta_S, \delta_F)}{\sigma_{\Delta F}^2} \\ &= \frac{R^{k-1} + R^{k-2} + ... + 1}{R^{2\cdot(k-1)} + R^{2\cdot(k-2)} + ... + R^2 + 1} \cdot h^* \end{aligned}$$

- We can see clearly that the previous optimal hedge ratio h^* is multiplied by what we call a hedge factor. The new hedge factor h^{**} is thus not really difficult to calculate.
- We will now explore the importance of the tail factor. We will do this by calculating the tail factor for different hedging horizons k. To do this we use an interest rate equal to 5% p.a. The daily gross rate R is defined as the final value minus the initial value as a percentage of the initial value. The results are summarized in the table below.

k	Tail Factor
10	0.9994
50	0.9967
100	0.993
1000	0.93

5.4.11 Implementation issues when hedging

- The optimal hedge ratio h^* is defined as:

$$h^* = \rho_{\Delta S,\Delta F} \cdot \frac{\sigma_{\Delta S}}{\sigma_{\Delta F}}$$

- To calculate this optimal hedge ratio h^*, we need to calculate the volatility of both the spot price changes and the forward price changes and the correlation between the forward price changes and the spot price changes.
- The question arises whether we need to calculate the volatilities and the correlation, based on price changes over the hedging horizon. Suppose for example that we want to calculate the optimal hedge ratio for a three month hedge. Do we then need the three-month changes of the spot price to calculate the volatility of the spot price changes? If we assume that the daily price changes are independent, the answer is negative. In that case, the variance of a sum is the sum of the variances. Mathematically, this becomes:

$$\begin{aligned}
\sigma^2_{\Delta S} &= k \cdot \sigma^2_{\delta S} \\
\rho_{\Delta S,\Delta F} &= \frac{cov(\Delta F, \Delta S)}{\sigma_{\Delta S} \cdot \sigma_{\Delta F}} \\
&= \frac{k \cdot cov(\delta S, \delta F)}{k \cdot \sqrt{\sigma^2_{\delta F} \cdot \sigma^2_{\delta F}}} \\
&= \rho_{\delta S,\delta F}
\end{aligned}$$

5.4.12 Calculating the hedging coefficient using linear regression

- We can calculate the hedging coefficient using a simple, linear regression.
  - The simple linear regression model and the corresponding estimation for the slope coefficient β are defined as:

$$\begin{aligned}
Y &= \alpha + \beta x + \epsilon \\
\beta &= \frac{cov(X, Y)}{var(X)}
\end{aligned}$$

  - We need to estimate the optimal hedge ratio h^* which is given by:

$$h^* = \frac{cov(\delta_s, \delta_F)}{var(\delta_F)}$$

  - We should therefore perform a simple linear regression on the following equation:

$$\delta_s = \alpha + h^* \delta_F + \epsilon$$

- In the case that multiple contracts are used to set up a hedge, we need to regress the following equation:

$$\delta_s = \alpha + h^*_{F1}\delta_{F1} + h^*_{F2}\delta_{F2} + ... + \epsilon$$

5.4.13 Hedging a portfolio of equities

- Cross hedging is typically used when hedging a portfolio of equities. This could be a stock index for example.
- Suppose the spot price of an underlying stock index S_{t_0} is equal to \$1000, the interest rate r is equal to 4%, the dividend yield on the index δ is equal to 1% and the time to maturity of the future contract τ is equal to four months. The value of the portfolio is equal to \$5.050.000. The beta of the portfolio is the regression coefficient of the portfolio returns on the market returns and is equal to 1.5. The contract size of a future contract on the index is equal to 250 times the index. We want to calculate how many contracts are needed to hedge the portfolio.
- We can easily calculate the futures price for the index:

$$\begin{aligned} F &= S_{t_0} \cdot e^{(r-\delta)\tau} \\ &= 1000 \cdot e^{(0.04-0.01)\cdot\frac{4}{12}} \\ &= \$1010.05 \end{aligned}$$

- We first consider a naive hedge. This would mean hedging the entire value of the portfolio.

$$\begin{aligned} n &= \frac{Q}{H} \\ &= \frac{5.050.000}{250 \cdot 1010.05} \\ &= 20 \end{aligned}$$

- Now we want to determine the number of contracts that are needed in case of an optimal hedge. The optimal number of contracts is given by:

$$\begin{aligned} n^* &= h^* \cdot naive\ hedge \\ &= 1.5 \cdot 20 \\ &= 30 \end{aligned}$$

- Suppose the spot price of the index S_{t_0} evolves to \$900 within three months time.
  - The forward price of the index $F_{t_0+3m,T}$ would then evolve to:

$$\begin{aligned} F_{t_0+3m,T} &= 900 \cdot e^{(0.04-0.01)\cdot\frac{1}{12}} \\ &= 902.25 \end{aligned}$$

– In that case, the value of the position at that time is equal to:

$$\begin{aligned} f_{t_0+3m} &= 30 \cdot (1010, 05 - 902.25) \cdot 250 \\ &= \$808.500 \end{aligned}$$

- The expected return on the market is given by:

$$\begin{aligned} \gamma_{market} &= \frac{S_{t_0}}{F_{t_0,T}} - 1 + \delta\tau \\ &= \frac{900}{1000} - 1 + \frac{0.01}{4} \\ &= 0.9 - 1 + 0.0025 \\ &= -0.0975 \end{aligned}$$

- The expected return on the portfolio is given by:

$$\begin{aligned} \gamma_{portfolio} &= r\tau + \beta_{portfolio} \cdot (\gamma_{market} - \delta\tau) \\ &= \frac{0.04}{4} + 1.5 \cdot (-0.0975 - 0.01) \\ &= -0.1525\% \end{aligned}$$

- When the market goes down with 9.75%, our portfolio is expected to go down with 15.125%. In that case , the value of our portfolio becomes:

$$\begin{aligned} V(portfolio) &= 5.050.000 \cdot (1 - 0.1525) \\ &= \$4.286.187 \end{aligned}$$

- The total change in the value of the portfolio is thus given by:

$$\begin{aligned} \Delta(V(portfolio)) &= 5.050.000 - 4.286.187 \\ &= \$5096.187 \end{aligned}$$

5.4.14 Changing the beta of a portfolio

- In the previous example, we reduced the beta of the portfolio from 1.5 to zero by shorting a number of forward contracts. In general, we can set a target value for the beta of the portfolio. We change the value of the beta of our portfolio from the actual beta $\beta_{portfolio}$ to a target value for the beta of the portfolio β_{target} by shorting the following number of forward contracts:

$$n = (\beta_{portfolio} - \beta_{target}) \cdot \frac{portfolio\ value}{value\ of\ one\ future\ contract}$$

- Setting the target above the current beta for the portfolio results in a negative number, which means we have to take a long position in the resulting number of forward contract. Adding long future contracts to the portfolio would result in an increase in the beta of our portfolio.

6 Interest rate futures and forwards

We now take a look at interest rates derivatives. An interest rate derivative is a financial instrument with a value that is linked to the movements of an interest rate or rates. Examples of such derivatives are forward rate agreements, Eurodollar futures and bond futures. First, we take a look at the concepts of forward investing and forward borrowing. Afterwards, we examine forward rate agreements, Eurodollar futures and bond futures.

6.1 Forward investing and forward borrowing

6.1.1 Forward investing

Suppose we want to enter into an investment that starts at a future point in time t_1 and ends at maturity t_2. We are committing ourselves to invest a certain amount, over a future period. We can easily implement such an investment by taking the following steps:

1. Investing \$100 at time t_0 for a period of $(t_2 - t_0)$ at the current spot rate which is 4% p.a. c.c.
2. Borrowing \$100 at time t_0 at the prevailing interest rate of 3% p.a. c.c.

This creates a synthetic forward investment. We can now derive the interest rate that is received for such an investment. The interest rate over the second period r_{p_2} is equal to 5% p.a. c.c. This interest rate is defined as the forward rate.

$$
\begin{aligned}
100 \cdot e^{0.04 \cdot 2} &= -100 \cdot e^{0.03 \cdot 1} \cdot e^{r_{p_2} \cdot 1} \\
108.33 &= 103.05 \cdot e^{r_{p_2} \cdot 1} \\
ln(\frac{108.33}{103.05}) &= r_{p_2} \\
r_{p_2} &= 0.05
\end{aligned}
$$

6.1.2 Forward borrowing

Suppose we want to enter into a loan to borrow a certain amount of money starting at a future point in time t_3 and ending at maturity t_4. We can easily implement such a loan by taking the following steps:

1. Taking out a loan of \$100 for a period $(t_4 - t_0)$, at an interest rate of 5% p.a. c.c.
2. Investing \$100 at time t_0 for a period of $(t_3 - t_0)$ at an interest rate of 4.6% p.a. c.c.

This creates a synthetic forward loan. We van now derive the interest rate that is paid for such a loan. The interest rate over the fourth period r_{p_4} is equal to 6.2% p.a. c.c. This interest rate is defined as the forward rate.

$$
\begin{aligned}
100 \cdot e^{0.05 \cdot 4} &= 100 \cdot e^{0.046 \cdot 3} \cdot e^{r_{p_4} \cdot 1} \\
e^{0.200} &= e^{0.138 + r_{p_4}} \\
r_{p_4} &= 0.062
\end{aligned}
$$

6.2 The forward rate agreement

Forward rate agreements are over-the-counter contracts between parties that determine the rate of interest to be paid on an agreed-upon date in the future. In other words, a forward rate agreement is an agreement to exchange an interest rate commitment on a notional amount. A forward rate agreement is therefore an interest rate derivative. The notional amount is not exchanged, but rather a cash amount based on the interest rate differentials and the notional value of the contract.

We now list some important properties and conventions with regard to forward rate agreements:

- The long position in a forward rate agreement gains when the underlying interest rate increases in value. The underlying reasoning is that the long agrees to borrow the notional at the forward interest rate. When the spot interest rate increases, the long will have the opportunity to borrow at a lower interest rate. When the spot interest rate decreases, the long will be obligated to borrow at a higher interest rate. In practice however, the notional is not exchanged.
- The short position in a forward rate agreement gains when the underlying interest rate decreases in value. The underlying reasoning is that the short agrees to lend the notional at the forward interest rate. When the spot interest rate increases, the short be obligated to lend the notional at the lower, contracted interest rate. When the spot interest rate decreases, the short will have the opportunity to lend the money at the higher, contracted interest rate. In practice however, the notional is not exchanged.

- The parties that enter a forward rate agreement determine an interest rate that has to be paid in the future. The contracted interest rate is compared to the spot interest rate at a future point in time t_1. This difference is then multiplied with the notional amount as determined by the forward contract. This product constitutes the payoff of the forward contract at maturity. However, in the case of the forward rate agreement, the payoff is calculated at the maturity of the contract t_2. The payoff itself however still takes place at t_1. Therefore, the payoff is discounted over the period $(t_2 - t_1)$.
- The period $(t_1 - t_0)$ is called the deferment period.
- The period $(t_2 - t_1)$ is called the contract period.
- The dealing date t_0 is the date on which the contract rate is agreed upon. On this date, the trade occurs. This is the time of the inception of the contract.
- The fixing date t_1 is the date on which the reference rate is determined. This is the data on which the deferment period and the contract period are decoupled.
- The settlement date is the date on which the payments take place. This is usually the fixing date t_1.
- The maturity date t_2 is the date on which the forward rate agreement ends. This is also the data on which the payoff is calculated.
- The fixing data and the settlement data are explicitly mentioned in the notation for the forward rate contract. We typically talk about $(t_1 \times t_2)$ forward rate agreement.
- The contract rate is the interest rate that is fixed at the time of the inception of the forward contract t_0.
- The reference rate is the interest rate that is determined at the fixing date t_1. The reference rate is compared to the contract rate to determine the payoff of the forward contract.

6.2.1 Cashflows of a forward rate agreement

We already know that the contract rate is the interest rate that is fixed at the time of the inception of the forward contract t_0. The reference rate was the the interest rate that is determined at the fixing date t_1. The contract rate is compared to the reference rate to determine the payoff of forward contract.

However, the payoff itself is calculated at the maturity date of the forward contract t_2, id est at the end of the contract period t_1. In practice the payoff is paid out at the start the start of the contract period. The payoff as calculated at the end of the contract period is thus discounted back to the start of the contract period.

In general, the payoff of a long position in a forward rate agreement at the time of payoff t_1 is given by:

$$f_{t_1}^{long\ fra.} = \frac{(r_{t_1,t_2} - fr_{t_0,(t_1 \times t_2)}) \cdot \tau}{1 + r_{t_1,t_2} \cdot \tau}$$

Where:

- t_0 is the dealing data. This is the data on which the contract rate r_{t_0} is determined.
- t_1 is the fixing data. This is the date where the reference rate r_{t_1} is determined.
- t_2 is the maturity date. This is the date on which the contract rate r_{t_0} and the reference rate r_{t_1} are compared. It is therefore the date on which the payoff is calculated before it is discounted back to the fixing date t_1.
- τ is the contract period id est the period between the maturity date t_2 and the fixing date t_1.
- r_{t_1,t_2} is the interest rate at time t_1 for maturity t_2.
- $fr_{t_0,(t_1 \times t_2)}$ is the forward rate at time t_0 that applies for the period between t_1 and t_2.

6.2.2 The forward rate agreement as a mini-swap

- One can consider a forward rate agreement as a mini-swap. A swap is an exchange between two parties. At the time of the inception of the swap t_0 the exchanged underlying assets are equal in value. Subsequent cashflows at fixed points in time comprise the difference between the value of underlying borrowed at that time minus the value of the underlying lent at that time. Suppose the underlying that is borrowed has an interest rate r and the underlying that is lent has an interest rate s. Both interest rates are applied to a principal Pr.
- If for example that the value of both assets is compared at time t_1. The subsequent cashflow from the swap would be equal to:

$$\begin{aligned} f_{t_1}^{swap} &= (r_{t_1} - s_{t_1}) \cdot (t_1 - t_0) \cdot Pr. \\ &= r_{t_1} \cdot (t_1 - t_0) \cdot Pr. - s_{t_1} \cdot (t_1 - t_0) \cdot Pr. \end{aligned}$$

- It is clear from the equation above that a swap compares two cashflows. The principal idea behind a swap is to pay a fixed amount and receive a floating amount on the same principal or vice versa.

6.2.3 Pricing a new forward rate agreement

The fixed rate we contract today is the forward rate. The forward rate is the rate that is applicable to a new contract.

We can easily verify this. Suppose that at t_0, we enter into a long position of a $(t_1 \times t_2)$ forward contract. The payoff of the long position, at the start of the contract period is given by the following expression:

$$\begin{aligned} f_{t_1}^{long\ fra.} &= \frac{(r_{t_1,t_2} - fr_{t_0,(t_1 \times t_2)} \cdot \tau) \cdot P}{1 + r_{t_1,t_2} \cdot \tau} \\ &= \frac{P + P \cdot r_{t_1,t_2} \cdot \tau}{1 + r_{t_1,t_2} \cdot \tau} - \frac{P + P \cdot fr_{t_0,(t_1 \times t_2)} \cdot \tau}{1 + r_{t_1,t_2} \cdot \tau} \\ &= P - P \cdot \frac{1 + fr_{t_0,(t_1 \times t_2)} \cdot \tau}{1 + r_{t_1,t_2} \cdot \tau} \end{aligned}$$

The payoff of the forward rate agreement at time t_1 consists of two terms. The first term is a positive cashflow and is certain at the time of the inception of the forward contract t_0. This cashflow reflects borrowing the principal amount. The second term is uncertain at the time of the inception of the contract t_0. This is because the interest rate r_{t_1} in determined only at time t_1.

If we compound the second term up to time t_2 at interest rate r_{t_1}, the interest rate r_{t_1} disappears from the equation. Therefore, the second term is certain at time t_0 Mathematically:

$$\begin{aligned} & P \cdot \frac{1 + fr_{t_0,(t_1 \times t_2)} \cdot \tau}{1 + r_{t_1} \cdot \tau} \cdot (1 + r_{t_1} \cdot \tau) \\ & = P \cdot (1 + fr_{t_0,(t_1 \times t_2)} \cdot \tau) \end{aligned}$$

The present value at time t_0 of the payoff that takes place at time t_1 is therefore given by:

$$f_{t_0}^{long\ fra.} = P \cdot df_{t_1} - P \cdot (1 + fr_{t_0,(t_1 \times t_2)} \cdot \tau) \cdot df_{t_2}$$

We know that at the time of the inception of the forward contract, the forward contract is of no value to both parties. Therefore we can equate the expression above to zero and derive an expression for the interest rate that is contracted at t_0 i.e. $fr_{t_0,(t_1 \times t_2)}$. Mathematically this becomes:

$$\begin{aligned} f_{t_0}^{long\ fra.} &= P \cdot df_{\tau=1} - P \cdot (1 + fr_{t_0,(t_1 \times t_2)} \tau) \cdot df_{\tau=2} \\ 0 &= P \cdot df_{\tau=1} - P \cdot (1 + fr_{t_0,(t_1 \times t_2)} \tau) \cdot df_{\tau=2} \\ fr_{t_0,(t_1 \times t_2)} &= \frac{df_{\tau=1} - df_{\tau=2}}{df_{\tau=2}} \cdot \frac{1}{\tau} \end{aligned}$$

6.2.4 Example

Suppose the current 3-month Libor rate is equal to 4%. The current 6-month Libor rate is equal to 4.5%. There are 92 days in the first three-month period and 91 days in the second three-month period. We want to determine the forward interest rate for a (3×6) forward rate agreement. This means that the deferment period is equal to 3 months while the contract period is equal to 6 months.

First, we need to determine the discount factors for periods of three and six months. Mathematically, this becomes:

$$\begin{aligned} df_{\tau=3m} &= \frac{1}{1 + r_{(t_0,\, t_0+3m)} \cdot \tau} \\ &= \frac{1}{1 + 0.04 \cdot \frac{92}{360}} \\ &= 0.98988 \end{aligned}$$

$$\begin{aligned} df_{\tau=6m} &= \frac{1}{1 + r_{(t_0,\, t_0+6m)} \cdot \tau} \\ &= \frac{1}{1 + 0.045 \cdot \frac{183}{360}} \\ &= 0.97764 \end{aligned}$$

We can then calculate the forward rate:

$$\begin{aligned} fr_{t_0,(t_1 \times t_2)} &= \frac{df_{\tau=3m} - df_{\tau=6m}}{df_{\tau=6m}} \cdot \frac{1}{\tau} \\ &= \frac{0.98988 - 0.97764}{0.97764} \cdot \frac{360}{91} \\ &= 0.0495 \end{aligned}$$

Notice that we can also calculate the forward rate in a different way. The interest payed on a loan with a maturity of six months has to be equal to a loan with a maturity of three months combined with a loan with a maturity of three months that starts three months from now. Mathematically, this becomes:

$$\begin{aligned} 1 + r_{(t_0,\, t_0+6m)} \cdot \tau &= (1 + r_{(t_0,\, t_0+3m)} \cdot \tau) \cdot (1 + fr_{t_0,(t_1 \times t_2)}) \\ 1 + 0.045 \cdot \frac{183}{360} &= (1 + 0.04 \cdot \frac{92}{360}) \cdot (1 + fr_{t_0,(t_1 \times t_2)}) \\ fr_{t_0,(t_1 \times t_2)} &= 0.0495 \end{aligned}$$

6.2.5 Pricing an existing forward rate agreement

To price a forward rate agreement at a random point in time during its lifetime, we need to compute the value of the cashflows of the contract at that point in time. This corresponds to adjusting the discount factors.

6.2.6 Example

Suppose the current 3-month Libor rate is equal to 4%. The current 6-month Libor rate is equal to 4.5%. There are 92 days in the first three-month period and 91 days in the second three-month period. We want to determine the value of a (3×6) forward rate agreement, after one month.

The two-month Libor rate at that time is equal to 5.5%. The five-month Libor rate at that time is equal to 6.0%.

The time until the maturity of the deferment period is equal to 61 days. The time until the maturity of the contract period is equal to 91 days.

Calculate the discount factors:

$$\begin{aligned} df_{\tau=2m} &= \frac{1}{1 + r_{(t_0,\, t_0+2m)} \cdot \tau} \\ &= \frac{1}{1 + 0.055 \cdot \frac{61}{360}} \\ &= 0.99077 \end{aligned}$$

$$\begin{aligned} df_{\tau=5m} &= \frac{1}{1 + r_{(t_0,\, t_0+5m)} \cdot \tau} \\ &= \frac{1}{1 + 0.060 \cdot \frac{152}{360}} \\ &= 0.97529 \end{aligned}$$

With this we can calculate the actual forward rate:

$$\begin{aligned} fr_{t_0+1m,(t_1 \times t_2)} &= \frac{df_{\tau=2m} - df_{\tau=5m}}{df_{\tau=5m}} \cdot \frac{1}{\tau_{3m}} \\ &= \frac{0.99077 - 0.97529}{0.97529} \cdot \frac{360}{91} \\ &= 0.062791129 \end{aligned}$$

We can calculate the value of the long forward rate agreement at time t_2 as the difference between the new and the old forward rate multiplied with the time span of the contract period. We can then discount this value back to the current point in time $t_0 + 1m$. Mathematically, this becomes:

$$\begin{aligned}
f_{t_0+1m} =& (fr_{t_0+1m,(t_1 \times t_2)} - fr_{t_0,(t_1 \times t_2)}) \cdot P \cdot \tau \cdot df_{\tau=5m} \\
&= (0.062791129 - 0.0496) \cdot \$25.000.000 \cdot \frac{91}{360} \cdot 0.97529 \\
&= 0.013191129 \cdot \$25.000.000 \cdot 0.252777778 \cdot 0.97529 \\
&= \$81300.77
\end{aligned}$$

Thanks to the forward rate agreement, we have locked in the interest rate at 4.5%. The term structure went up after one month. This means the forward rate at the time of valuation is higher. Therefore, the long position gains in value.

6.3 Hedging with forward rate agreements

A long hedge is meant to fix an interest rate when borrowing in a future time period. To set up a long hedge, the hedger enters a long forward rate agreement. The deferment period is equal to the time until the party enters the loan. The contract period starts at this time and ends at the maturity of the loan. The position in the forward rate agreement increases in value when the interest rate increases during the deferment period. However, this gain will be offset by entering the loan at at this higher interest rate at the end of the deferment period. The effective borrowing cost will be fixed at the forward rate $fr_{t_0,(t_1 \times t_2)}$.

At time t_1 i.e. at the end of the deferment period, the contracted forward rate will be compared against the spot rate r_{t_1} at that time. The cashflow that results from the forward rate agreement is equal to:

$$f_{t_1}^{fra.} = (r_{t_1,t_2} - fr_{t_0,(t_1 \times t_2)}) \cdot df_{\tau_2}$$

This cashflow will be offset by the gain or loss in the spot market. Suppose the interest on the loan is paid at the maturity of the loan. The total cashflow of the hedge at time t_2 then becomes:

$$\begin{aligned}
f_{t_2}^{long\ hedge} &= (r_{t_1,t_2} - fr_{t_0,(t_1 \times t_2)}) - r_{t_1,t_2} \\
&= -fr_{t_0,(t_1 \times t_2)}
\end{aligned}$$

The hedger ends up paying the contracted forward rate.

A short hedge is meant to fix an interest rate when lending in a future period. To set up a short hedge, the hedger enters into a short forward rate agreement. The position in the forward rate agreement increases in value when the interest rate decreases during the deferment period. However, this gain will be offset by entering the loan at at this lower interest rate at the end of the deferment period. The effective borrowing cost will be fixed at the forward rate $fr_{t_0,(t_1 \times t_2)}$.

At time t_1 i.e. at the end of the deferment period, the contracted forward rate will be compared against the spot rate r_{t_1,t_2} at that time. The cashflow that results from the forward rate agreement is equal to:

$$f_{t_1}^{fra.} = (fr_{t_0,(t_1 \times t_2)} - r_{t_1,t_2}) \cdot df_{\tau_2}$$

This cashflow will be offset by the gain or loss in the spot market. Suppose the interest on the loan is paid at the maturity of the loan. The total cashflow of the hedge at time t_2 then becomes:

$$\begin{aligned} f_{t_1}^{long\ hedge} &= (fr_{t_0,(t_1 \times t_2)} - r_{t_1,t_2}) + r_{t_1,t_2} \\ &= fr_{t_0,(t_1 \times t_2)} \end{aligned}$$

The hedger ends up receiving the contracted forward rate in all scenarios.

6.4 Eurodollar futures

A Eurodollar future is the exchange-traded equivalent of a forward rate agreement. Futures are exchange traded, quoted on a daily basis and marked-to market.

The quotation of Eurodollar futures is based on a futures price id est, the interest rate is not quoted in itself. However, we can still derive the interest rate from the quoted futures price. The quoted future price is a function of the interest rate. This makes the Eurodollar future somewhat different from the forward rate agreement.

We will also not have to do the discounting that took place in the forward rate agreement. This is because the contract is settled at the beginning of the contract period. Id est the settlement is calculated at the beginning of the contract period not at the end of the contract period.

6.4.1 Eurodollar deposits

A Eurodollar refers to US Dollars that are held outside of the USA. A Eurodollar is thus a Dollar that is not in the USA. The applicable interest rate is the Libor. The most popular maturities for Eurodollar deposits are three and six months. The day counting conventions depends on the market:

- For the United States, Japan and Europe $\frac{ACT}{360}$ is used.
- For Great Britain, Canada and Australia $\frac{ACT}{365}$ is used.

Consider the following example. A three month deposit of \$1.000.000 has a lifespan from from March 16th to June 15th and yields an interest rate of 4% p.a. a.c. This Eurodollar deposit will yield the following amount of interest, at its maturity:

$$I = \$1.000.000 \cdot 0.04 \cdot \frac{91}{360} = \$10111.11$$

6.4.2 Eurodollar futures characteristics

We now list some important properties and conventions with regard to forward rate agreements:

- The notional value of a Eurodollar future contract is equal to \$1.000.000.
- The underlying is a non-tradable time deposit.
- The deferment period can be different from contract to contract.
- The contract period is always three months.
- The last trading day is always two days before the third Wednesday of the delivery month.
- The future is settled daily. On the last trading day, the final settlement takes place.

Because Eurodollar futures are so standardized, we can easily calculate the cost of a tick. Id est the change in the value of the Eurodollar future after an increase or decrease of one basis point in the underlying. A decrease of one basis point in the 3-month Libor results in a decrease in the value of the long position in the Eurodollar future. This loss is given by:

$$\begin{aligned}\Delta f^{\,long\ edf.} &= \$1.000.000 \cdot 1\ bp. \cdot \frac{90}{360} \\ &= \$1.000.000 \cdot 0.0001 \cdot \frac{90}{360} \\ &= \$25\end{aligned}$$

The minimum price move is $\frac{1}{4}$ of a tick i.e. \$6.25 for contracts in the expiration month and $\frac{1}{2}$ of a tick i.e. \$12.50 for all other contracts.

6.4.3 Quotation for Eurodollar futures

The quotation for Eurodollar futures is a function of the underlying i.e. the Libor interest rate. The quote for a Eurodollar future is defined in the following way:

$$QP = 100 - 100 \cdot L$$

With L being the 3-month Libor with quarterly compounding.

When the Libor increases in value, the future quote goes down. This means the long position gains when the quoted future price declines and loses when the quoted future price increases. The short party gains when the future price increases and loses when the future price declines.

6.4.4 Payoff of a Eurodollar future

We express the payoff of the Eurodollar future contract in terms of basis point changes in the underlying, being the Libor r_{t_1,t_2}. Note that we keep using the same notation as with the interest rate forward contract for our convenience.

$$f_{t_1}^{short\ edf.} = \$25 \cdot (fr_{t_0,(t_1 \times t_2)} - r_{t_1,t_2}) \cdot 10.000$$

6.4.5 Hedging with Eurodollar future contracts

Consider the following example. An investor anticipates a three-month investment opportunity, starting in September with a notional value of \$100.000.000. The investment provides the prevailing Libor rate at the start of the investment period. The investor wants to bridge the period until the start of the investment. To do so he wants to lock in the current three month interest rate for a period of three months. This can be done by entering a long Eurodollar future contract that expires in September. Suppose the quote for the Eurodollar future is equal to \$96.50. This indicates that the investor can lock in an interest rate equal to:

$$\begin{aligned} L &= \frac{\$100 - \$96.50}{\$100} \\ &= 0.035 \end{aligned}$$

To lock in the interest rate, the investor enters 100 long forward contracts:

$$N = \frac{\$100.000.000}{\$1.000.000} = 100$$

Suppose that, two days before the third Wednesday of September, the 3-month Eurodollar rate turns out to be 2.6%. The final settlement is then at a price of \$97.40. The payoff from the Eurodollar future is given by:

$$\begin{aligned} f_{t_1}^{long\ edf.} &= (\$25 \cdot (fr_{t_0,(t_1 \times t_2)} - r_{t_1,t_2}) \cdot 10.000) \cdot N \\ &= (\$25 \cdot (0.035 - 0.026) \cdot 10.000) \cdot 100 \\ &= \$225.000 \end{aligned}$$

The interest, earned on the 3-month investment is therefore equal to:

$$\begin{aligned} I &= Pr \cdot \tau \cdot r_{t_1,t_2} \\ &= 100.000.000 \cdot 0.25 \cdot 0.026 \\ &= \$650.000 \end{aligned}$$

The gain on the Eurodollar futures, brings the cumulative amount of the cashflows up to:

$$\begin{aligned} f_{t_1}^{long\ hedge} &= I + f_{t_1}^{long\ edf.} \\ &= \$650.000 + \$225.000 = \$875.000 \end{aligned}$$

We can see easily that this amounts to the cashflow that results from investing the principal, at the interest rate at the time of entering the Eurodollar futures t_0. Id est we locked in the interest rate that applied at the time of entering the Eurodollar future contracts.

$$\begin{aligned} f_{t_1}^{long\ hedge} &= fr_{t_0,(t_1 \times t_2)} \cdot Pr \cdot \tau \\ &= 0.035 \cdot \$100.000.000 \cdot 0.25 \\ &= \$875000 \end{aligned}$$

6.4.6 Complications real world hedging

In practice, there are some complications when hedging with Eurodollar future contracts:

1. The contract period will not allays be exactly equal to 90 days.
2. The timing of the cashflows is not aligned. Id est the future contract will come with a cashflow at the end of the deferment period t_1. However, an investment or a loan comes with cashflows at the end of the contract period. We want to compare both cashflows at the same point in time. Therefore, we need to compound the payoff of the Eurodollar future, that takes place at the end of the deferment period t_1, to the maturity of the investment t_2. The payoff of the hedge is then given by:

   $$f_{t_1}^{long\ hedge} = 1.000.000 \cdot r_{t_1,t_2} \cdot \tau + \left[25 \cdot (fr_{t_0,(t_1 \times t_2)} - r_{t_1,t_2}) \cdot 10000\right] \cdot (1 + r_{t_1,t_2} \cdot \tau)$$

   We can easily see that our original position in the Eurodollar future is too large due to the compounding factor. The solution to this exists in reducing the magnitude of the original position with a factor equal to:

   $$df = \frac{1}{1 + \tau \cdot r_{t_1,t_2}}$$

   However, we are setting up the hedge at time t_0. At this time, we do not know what the Libor rate at the end of the deferment period r_{t_1,t_2} is going to be. We therefore have to predict a value for r_{t_1,t_2}. Often, the forward rate is taken to be a good estimate for the future spot rate. Mathematically, this becomes:

   $$fr_{t_0,(t_1 \times t_2)} \approx r_{t_1,t_2}$$

6.4.7 Tailing the hedge

We should not forget that the daily settlements have an impact on the optimal size of the hedge. We now try to adjust the hedge size, taking into account the impact of daily settlements.

Consider a short position in a Eurodollar future to hedge the interest rate risk on a loan in the future. The loan has a time to maturity of 90 days. An increase of α basis points in the Libor would:

1. Give an immediate cash inflow on the future contract, equal to:

$$\Delta f^{\,long\ edf.} = \alpha \cdot 25$$

2. Increase the interest burden on the loan with:

$$\Delta I_{t_2} = 1.000.000 \cdot 0.0001 \cdot \tau$$

   However, this cashflow takes place at the end of the maturity of the loan. We therefore need to discount this cashflow to the start of the contract period using the Libor rate that applies to the maturity of the loan $r_{t_0,T}$. The increase in the interest rate burden therefore becomes:

$$\Delta I_{t_1} = \frac{1.000.000 \cdot 0.0001 \cdot \tau}{1 + r_{t_0,T} \cdot \tau}$$

3. We now need to choose α in such a way, that the advantage from the Eurodollar future and the disadvantage from the increase in the interest burden, cancel each other out. Mathematically, this becomes:

$$\begin{aligned}
\Delta f^{\,long\ edf.} &= \Delta I_{t_1} \\
\alpha \cdot 25 &= \frac{1.000.000 \cdot 0.0001 \cdot \tau}{1 + r_{t_0,T} \cdot \tau} \\
\alpha &= \frac{1.000.000 \cdot 0.0001 \cdot \tau}{25 \cdot (1 + r_{t_0,T}) \cdot \tau} \\
&= \frac{1}{1 + r_{t_0,T} \cdot \tau} \\
&= \frac{1}{1 + r_{t_0,T} \cdot \frac{90}{360}}
\end{aligned}$$

6.5 Comparing the forward rate agreement and the Eurodollar future

We now want to compare the forward rate of forward rate agreements with the future rate of Eurodollar futures.

The payoff of the long forward rate agreement at the end of the deferment period t_1, is given by:

$$f_{t_1}^{long\ fra.} = \frac{\left(r_{t_1,t_2} - fr_{t_0,(t_1 \times t_2)}\right) \cdot \tau}{1 + r_{t_1,t_2} \cdot \tau}$$

The payoff of a short position in a Eurodollar future at the end of the deferment period t_1, is given by:

$$\begin{aligned} f_{t_1}^{short\ edf..} &= \frac{1}{1 + fr_{t_0,(t_1 \times t_2)} \cdot \tau} \cdot \$25 \cdot 10.000 \cdot \left(r_{t_1,t_2} - fr_{t_0,(t_1 \times t_2)}\right) \\ &= 1.000.000 \cdot \frac{\left(r_{t_1,t_2} - fr_{t_0,(t_1 \times t_2)}\right) \cdot \tau}{1 + r_{t_0,T} \cdot \tau} \end{aligned}$$

We introduced the discount factor in the formula for the payoff of the forward rate agreement because the cashflows are calculated at time t_2 but take place at time t_1. This was not the case with the Eurodollar future contract. In the case of the Eurodollar future contract, the payment takes place and is calculated at time t_1. However, we introduced a discount factor for the payoff of the Eurodollar future to adjust the hedge size so it would take into account daily settlements.

6.6 Bond futures

In the case of the Eurodollar future and the interest rate forward, the underlying was a short term interest rate i.e. a money market interest rate. Now we will consider forward contracts where the underlying is a bond.

6.6.1 The quotation of bonds

When quoting a bond, we need to take into account the accrued interest since the last coupon date. The quoted price is always a percentage of the nominal value of the bond. The cash price is the quoted price plus the accrued interests, since the last coupon date.

6.6.2 Example

Consider a bond that expires $on 10/07/xx$ The most recent coupon payment took place at 10/01/xx and the next coupon date is $10/07/xx$. The day count convention is $\frac{act}{act}$. The bond price is quoted as $\$95 - 16$. The price, expressed in a Dollar amount is therefore given by:

$$\begin{aligned} P &= \$95 - \$\frac{16}{32} \\ &= \$95.50 \end{aligned}$$

We can now calculate the accrued interest since the last coupon date:

$$\begin{aligned} I_{accr.} &= \frac{54}{181} \cdot \$5.5 \\ &= \$1.64 \end{aligned}$$

We can now calculate the cash price for the bond. The cash price is given by:

$$\begin{aligned} CP &= \$95.5 + \$1.64 \\ &= \$97.14 \end{aligned}$$

6.6.3 Bond futures quotation

Future contracts are quoted in the same way as their underlying. For example, if bonds are quoted as a percentage of their nominal value, then the future contracts on this bond will also be quoted as a percentage of the nominal value of the bond.

As stated above, the quoted price does not take the accrued interest into account. We therefore have to correct the quoted price for accrued interests since the last coupon payment.

6.6.4 Treasury bond futures

We now consider a treasury bond future. The bond on which this future contract is written in a hypothetical bond i.e. the underlying bond does not exist in reality. There is of course a link between the deliverable bonds and this hypothetical bond. This link is made using so-called conversion factors. These conversion factors are calculated by the exchange. Remember that the short party has the delivery options and can thus choose which bond to deliver.

We denote the quote for the future on this hypothetical bond as Q_{fut}. Converting the future quote on the hypothetical bond to a future price for a deliverable bond, is done by multiplying the quote with a conversion factor cf for the deliverable bond. Mathematically:

$$P = Q_{fut} \cdot cf$$

This price however, will still need to be corrected for the accrued interests, since the last coupon payment. This corrected amount is the total cash that is received. Mathematically this becomes:

$$CP = Q_{fut} \cdot cf + I_{accr.}$$

6.6.5 Example

Suppose the future price is equal to \$90. The conversion factor cf for the deliverable bond is equal to 1.83. The accrued interest on the particular bond that we will be delivered is equal to \$3 per \$100. The cash price is theerfore equal to:

$$CP = \$90 \cdot 1.83 + \$3$$

The party that is short sells the bond and will receive the cash price. The party that is long and buys the bond will have to pay the cash price.

6.6.6 The impact of the delivery option

The short has to decide which true bond to deliver vis à vis the hypothetical bond. The short party has the delivery option and will therefore always choose the cheapest-deliverable bond. The short will maximize the net cashflow he receives. This net cashflow is equal to the cashflow he receives by selling the bond via the future contract minus the price he paid for that bond. Mathematically, this becomes:

$$\begin{aligned}
&= max\Big[(Q_{fut.} \cdot cf + I_{accr.}) - (Q_{bond\,price} + I_{accr.})\Big] \\
&= min\Big[Q_{bond\,price} - Q_{fut.} \cdot c\Big] \\
&= min\left[\frac{Q_{bond\,price}}{c}\right]
\end{aligned}$$

We can easily create a table for different bonds where we calculate for each bond the fraction above. We then choose the bond with the smallest fraction.

6.6.7 Example

The future value will depend on the value of the cheapest deliverable bond. Consider for example the following three bonds:

Bond price	c	$Q_b - Q_f$	Q_p/c
\$99.50	1.0582	2.69	95.84
\$143.50	1.5188	1.78	94.48
\$119.75	126.12	2.12	94.95

We can clearly see that the second bond is the cheapest-deliverable bond. The short party will prefer to deliver this bond over the other available bonds.

6.6.8 Bond future valuation

The value of the future depends on the cheapest-deliverable bond. In general, we move trough the following steps:

1. We determine the cheapest-deliverable bond.
2. We compute the quoted price in Dollar amounts.
3. We compute the accrued interests.
4. We determine the true value of the bond.
5. We can multiply the true value of the bond with the corresponding conversion factor, to acquire the future price for the hypothetical bond underlying the future contract.

Notice that we are therefore writing a future on the true value of this cheapest-deliverable bond. Therefore the future price is given by the following equation:

$$F_{t_0} = (S_{t_0} - I_{accr.}) \cdot e^{r \cdot (T - t_0)}$$

Where $I_{accr.}$ represents the present value of the accrued interests and S_{t_0} represents the quoted spot price of the bond in Dollar amounts.

Notice that the short party needs to determine which bond will be cheapest to deliver. However, he does not know which bond this will be. The short therefore has to predict which bond this will be. We will have to price the future as if the cheapest-deliverable bond will not change.

6.6.9 The bond price interest rate sensitivity

The theoretical price of a bond is given by the sum of the present value of all future cashflows. Mathematically:

$$P = e^{-s_1 \cdot \tau_1} \cdot CF_1 + ... + e^{-s_n \cdot \tau_n} \cdot CF_n$$

Consider a small shock to all spot rates equal to Δs. With this shock, the theoretical price of the bond changes to:

$$P = e^{(-s_1 + \Delta s) \cdot \tau_1} \cdot CF_1 + ... + e^{(-s_n + \Delta s) \cdot \tau_n} \cdot CF_n$$

We can now calculate the the different between the bond price after the shock in the interest rate and the price before the shock in the interest rate:

$$\begin{aligned} \Delta P &= \left[e^{-(s_1 + \Delta s) \cdot \tau_1} - e^{s_1 \tau_1}\right] \cdot CF_1 + ... + \left[e^{-(s_n + \Delta s) \cdot \tau_n} - e^{s_n \tau_n}\right] \cdot CF_n \\ &= \left[e^{-s_1 \tau_1} \cdot (e^{\Delta s \tau_1} - 1) \cdot CF_1 + ... + e^{-s_n \tau_n} \cdot (e^{\Delta s \tau_n} - 1) \cdot CF_n\right] \end{aligned}$$

We know that:

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + ... \\ &\approx 1 + x \end{aligned}$$

Applying this formula to the expression for ΔP, we get:

$$\begin{aligned} \Delta P &= e^{-s_1 \cdot \tau_1} \cdot (1 - \Delta s \cdot \tau_1 - 1) \cdot CF_1 + ... + e^{-s_n \cdot \tau_n} \cdot (1 - \Delta s \cdot \tau_n - 1) \cdot CF_n \\ &= -\left[e^{-s_1 \tau_1} \cdot CF_1 \cdot \tau_1 \cdot \Delta s + ... + e^{-s_n \tau_n} \cdot CF_n \cdot \tau_n \cdot \Delta s\right] \\ \frac{\Delta P}{P} &= -\left[\frac{e^{-s_1 \tau_1} \cdot CF_1}{P} \cdot \tau_1 + ... + \frac{e^{-s_n \tau_n} \cdot CF_n}{P} \cdot \tau_n\right] \cdot \Delta s \\ &= -(w_1 \tau_1 + ... + w_n \tau_n) \cdot \Delta s \\ &= -Duration \cdot \Delta s \end{aligned}$$

6.6.10 The bond future price interest sensitivity

We will now calculate the duration of a bond future.

$$
\begin{aligned}
F &= P_{t_0} \cdot e^{s_f \cdot \tau_F} \\
&= \left(CF_1 \cdot e^{-s_1 \tau_1} + ... + CF_2 \cdot e^{-s_2 \tau_2}\right) \cdot e^{s_f \tau_f} \\
&= \left(CF_1 \cdot e^{s_f \cdot \tau_f - s_1 \cdot \tau_1} + ... + CF_n \cdot e^{s_f \cdot \tau_f - s_n \cdot \tau_n}\right)
\end{aligned}
$$

We apply a shock to the interest rate and study the effect on the future price:

$$
F = CF_1 \cdot e^{(s_f + \Delta s) \cdot \tau_f - (s_1 + \Delta s) \cdot \tau_1} + ... + CF_n \cdot e^{(s_f + \Delta s) \cdot \tau_f - (s_n + \Delta s) \cdot \tau_n}\big)
$$

With this, we can calculate the difference between the bond price after the interest rate shock and the bond price before the interest rate shock.

$$
\begin{aligned}
\Delta F &= CF_1 \cdot \left(e^{(s_f + \Delta s) \cdot \tau_f - (s_1 + \Delta s) \cdot \tau_1} - e^{s_f \tau_f - s_1 \tau_1}\right) + ... \\
&= CF_1 \cdot \left(e^{s_f \tau_f - s_1 \tau_1} \cdot e^{(\tau_f - \tau_1) \Delta s} - e^{s_f \tau_f - s_1 \tau_1}\right) + ... \\
&= CF_1 \cdot \left(e^{(\tau_f - \tau_1) \cdot \Delta s} - 1\right) \cdot e^{s_f \tau_f - s_1 \tau_1} + ... \\
&= CF_1 \cdot \big((\tau_f - \tau_1) \cdot \Delta s\big) \cdot e^{s_f \tau_f - s_1 \tau_1} + ...
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Delta F}{F} &= \frac{CF_1 \cdot \big((\tau_f - \tau_1) \cdot \Delta s\big) \cdot e^{s_f \tau_f - s_1 \tau_1} + ...}{F} \\
&= \frac{CF_1 \cdot \big((\tau_f - \tau_1) \cdot \Delta s\big) \cdot e^{s_f \tau_f - s_1 \tau_1} + ...}{P_{t_0} \cdot e^{s_f \tau_f}} \\
&= \left[(\tau_f - \tau_1) \cdot \frac{CF_1 \cdot e^{s_f \tau_f - s_1 \tau_1}}{P_{t_0} \cdot e^{s_f \tau_f}} + ...\right] \cdot \Delta s \\
&= \Big[(\tau_f - \tau_1) \cdot w_1 + ...\Big] \cdot \Delta s
\end{aligned}
$$

The equation gives the percentual change in the future price due to a shock in the interest rate. This is clearly a negative relationship.

6.6.11 Hedging interest rate risk with bond futures

We can try to protect ourselves from these parallel shifts in the term structure of interest rates. We hedge against this interest rate risk by matching the duration of assets and liabilities.

$$\begin{aligned}\frac{\Delta P}{P} &= -D_s \cdot \Delta s \\ \Delta P &= -D_s \cdot \Delta s \cdot P\end{aligned}$$

$$\begin{aligned}\frac{\Delta F}{F} &= -D_f \cdot \Delta s \\ \Delta F &= -D_f \cdot \Delta s \cdot F\end{aligned}$$

We want to reduce compensate the price change of the bonds with the price change of the future contracts. Mathematically:

$$\Delta P - N \cdot \Delta F = 0$$

Hence:

$$\begin{aligned}N^* &= \frac{\Delta P}{\Delta F} \\ &= \frac{-D_s \cdot \Delta s \cdot P}{-D_f \cdot \Delta s \cdot F} \\ &= \frac{D_s \cdot P}{D_f \cdot F}\end{aligned}$$

There are some limitations to this approach however:

- We use small interest rate shock for deriving the duration formulas.
- We use uniform interest rate shocks. Id est the shock in the term structure of interest rates is parallel.
- We apply only one shock at a time.

7 Swaps

In this chapter, we will take a closer look at swap contracts. We provide a brief overview of the different subjects that are discussed within this chapter:

- Definition of a swap contract.
- Comparative advantage.
- Interest rate swaps.
  - Overview
  - The par yield.
  - The zero curve with swap rates.
  - Valuation.
  - Overnight indexed swaps.
- Currency swaps.
  - Overview.
  - Valuation.
- Other swaps.

7.1 Definition of a swap contract

A swap contract is an Over The Counter agreement i.e. it is not exchange traded. The contract is concluded between two institutional parties. The contract entails an exchange of future cashflows, according to a prearranged formula.

A forward contract can be viewed as a simple version of a swap. However, swaps typically lead to cashflow exchanges on several future dates. This is the also key difference between a forward contract and a swap.

Swaps alter the cashflows from assets or liabilities into preferred forms. That is, investors use swap contracts to transform assets and liabilities.

7.2 Interest rate swaps

Interest rate swaps are one of the most important categories of swaps. An plain vanilla interest rate swap is defined as a conversion of interest payments, based on a fixed rate into payments based on a floating rate or vice versa. One of the counterparties pays based on a fixed rate while the other party pays based on a floating rate. These payments are exchanged on multiple future points in time.

There are two parties involved in a swap contract. One of the parties makes payments based on a fixed rate that in applied to the notional. The other party makes payments based on a floating rate that is applied to the notional. The resulting cashflows are exchanged on multiple occasions in the future. The fixed interest rate is called the swap rate. The floating interest rate is often the Libor. The notional principal itself is not exchanged for interest rate swaps.

7.2.1 Example

Consider an example with two companies Intel and Microsoft. Microsoft pays a fixed interest rate while Intel pays the floating interest rate. The fixed interest rate is equal to 5% p.a. s.c. while the floating interest rate is equal to the 6-month Libor. The notional amount is equal to \$100.000.000. The length of the contract is equal to five years. The cashflow exchanges take place semi-annually.

The table below gives an overview of the different cashflows throughout the life of the swap contract. Note that there is no uncertainty with regards to the first cashflow.

Date	Libor	Floating cf.	Fixed cf.	Nett cf.
0 years	4.20			
0.5 years	4.80	+2.10	-2.50	-0.40
1.0 years	5.30	+2.40	-2.50	-0.10
1.5 years	5.50	+2.65	-2.50	+0.15
2.0 years	5.60	+2.75	-2.50	+0.25
2.5 years	5.90	+2.80	-2.50	+0.30
3 years		+2.95	-2.50	+0.45

7.2.2 Applications of interest rate futures

- Companies use interest rate futures to transform fixed rate assets/liabilities into floating rate assets/liabilities and vice versa. Consider the following example:
  - Microsoft takes out a loan with a nominal value of $100.000.000 at Libor +0.1% from outside lenders. Intel also takes out a loan from outside lenders with a nominal value of $100.000.000 and pays an interest rate of 5.2% fixed. Suppose Microsoft would rather pay a fixed rate and Intel would rather pay a floating rate.
  - The solution for both parties is to enter into a swap contract. The swap contract would determine that Microsoft pays the 5% fixed interest rate to Intel. Intel on the other hand would pay Libor. The result for Microsoft would be to pay a fixed interest rate of 5.1%. The result for Intel would be to pay the floating interest rate of Libor +0.2%
  - We can clearly see that swap contracts allow companies to transform floating rate liabilities/assets into fixed rate liabilities/assets.
- The main purpose of transforming assets and liabilities between fixed and floating rates is to hedge away interest rate risks. To see this, consider the following example:
  - A bank provides 20-year mortgage loans at a fixed rate of 3%. The bank funds itself by taking on deposits, which provide the floating rate of six month Libor plus 0.5%. It is clear that bank's assets provide a fixed rate while the bank liabilities are on a floating rate. This difference constitutes an interest rate risk. When interest rates rise, the outgoing cashflows will increase in value while the incoming cashflows will remain constant for foreseeable years. Therefore, if the Libor increases, the financing cost will also increase. The opposite also holds true.
  - A bank wants to hedge away this exposure to interest rates. To do this, the bank could enter into a swap contract. Consider for example that the bank enters a forward contract where the bank pays a fixed interest rate of 3% and where it receives a floating rate of Libor +2.5%. The result would be that the bank receives a floating rate of 2%.
  - After entering the swap contract, the bank will have hedged away its exposure to interest rate changes completely.

- Another reason to transform assets/liabilities is because the parties that enter the swap contract have a comparative advantage. To understand what this means, we go over the theory of the comparative advantage. This theory shows that trade is not a zero-sum game, when participants have a comparative advantage over one another.
  - Consider two different countries for instance China and the USA. Both countries produce cars and trucks. The number of cars that can be produced per unit of input is given in the table below.

Country	USA	China
Cars	40	10
Trucks	40	30

  - We can also make a graphical representation of the combinations of products that are producible for each country. This graph is called the production possibility frontier.

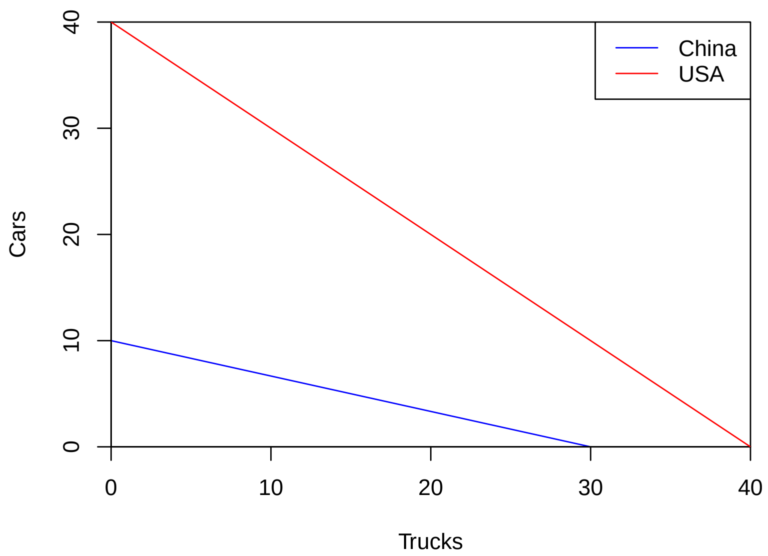


Figure 7.1: Production possibility frontier China and USA.

  - We can easily see that the USA has an absolute advantage in the production of both cars and trucks. We now ask ourselves whether both countries can set-up a system of trade that is beneficial to both.
  - To set up such a system we have to look at the opportunity cost of both cars and truck for both countries. For the USA, the opportunity cost of producing one car is equal to one truck. For China, the opportunity cost of producing one car is equal to three trucks. The opportunity cost for producing trucks is lower for China than for the USA. From this it becomes clear that China should specialize in producing trucks.
  - A system of trade that is beneficial for both nations can now be set up. China would be willing to give up two trucks for one car because its opportunity

cost to produce a car is higher at three trucks. The united states would be willing to give up one car for two trucks because its opportunity cost to produce a truck is higher at one car.

- Suppose the USA would trade ten cars for twenty trucks. In that case, the USA would have 30 cars and 20 trucks while China would have ten cars and ten trucks. We can easily see that these points lay on the right of the production possibility frontier. Id est these production combinations could not be reached before. Trading actually broadens the possibilities for both nations.
- Long story short; even tough one party has an absolute advantage in the production of all products, trade can be mutually beneficial. This is the case when both parties have a comparative advantage in one of the goods.

7.2.3 Swaps and the comparative advantage

- Companies transform assets and liabilities to take advantage of their comparative advantage in the interest rate market by entering a swap agreement.
- Consider the example where we have a first company called AAACorp and a second company called BBBCorp. AAACorp is interested in paying a floating rate while company B is interested in paying a fixed rate. Both companies can take out loans at the interest rates that are given in the table below.

Company	Fixed rate	Floating rate
AAACorp	4%	$6 - month\ Libor\ - 0.1\%$
BBBCorp	5.2%	$6 - month\ Libor\ + 0.6\%$

- AAACorp has an absolute advantage in both fixed and floating rates. However, it is clear that the companies each have a comparative advantage. The opportunity cost for the fixed interest rate is lower for BBBCorp than for AAACorp.
- AAACorp should borrow at the fixed rate while BBBCorp should borrow at the floating rate. Both parties can then set up a mutually beneficial swap agreement where AAAcorp pays a floating rate and receives a fixed rate and where BBBCorp takes the opposite position.
- Suppose for example that AAACorp enters into a loan where it pays a fixed interest rate of 4%. Company B enter into a loan where it pays a floating interest rate of $6 - month\ Libor\ + 0.6\%$. Both parties conclude a swap agreement where AAACorp would pay Libor and would receive a fixed interest rate of 4.35%. AAACorp would then end up paying $6 - month\ Libor\ - 0.35\%$. BBBCorp would end up paying a fixed interest rate of 4.95%. Both parties benefit. Id est, both parties end up paying interest rates that are lower than they could have been able to contract themselves.
- Notice that in this case, the advantage for both parties is equal. However, this does not need to be the case.
- There is no comparative advantage when the opportunity cost for both parties is equal. In that case, the production possibility frontiers are parallel.

7.2.4 Finding a counterparty

- We now ask ourselves how a company can find another company that would be interested in taking the opposite position in a swap agreement. Id est, how does a company find a suitable counterparty?
- In practice, large financial institutions are willing to act as market makers. Market makers are prepared to enter into a swap without having an offsetting position within another swap. The financial institution charges a spread for this service. This spread is usually equal to three to four basis points.
- These market makers quote bid/ask rates. These quoted rates are the fixed rates the financial institution is willing to exchange for a given floating rate like Libor. The quoted bid interest rate is the fixed interest rate a company will receive for paying the Libor. The quoted ask interest rate is the fixed interest rate a company will have to pay to receive the Libor.

7.2.5 The par yield

- The par yield is the coupon rate that causes the price of a bond to be equal to its par/principle value.
- We are looking for the coupon rate, given the term structure of interest rates for which the bond price is equal to its principal value.
- Consider the example of a two-year bond with annual coupons and a par value of \$100. The term structure of interest rates is given in the table below.

Time to maturity	Zero rate
1 year	4%
2 years	6%

We can then easily calculate the coupon for the bond:

$$\begin{aligned} P &= PV(coupon) + pv(coupon + principal) \\ P &= C_{t_1} \cdot e^{-r_{t_0,t_1} \cdot \tau_1} + C_{t_2} \cdot e^{-r_{t_0,t_2} \cdot \tau_2} + Pr \cdot e^{-r_{t_0,t_2} \cdot \tau_2} \\ 100 &= C \cdot e^{-0.04 \cdot 1} + C \cdot e^{-0.06 \cdot 2} + 100 \cdot e^{-0.06 \cdot 2} \\ C &= 6.12 \end{aligned}$$

From this, we can easily derive the coupon rate as follows:

$$\begin{aligned} \gamma &= \frac{C}{Pr} \\ &= \frac{6.12}{100} \\ &= 0.0612 \end{aligned}$$

7.2.6 Valuation of interest rate swaps

- To calculate the value of an interest rate swap we should take the sum of the present value of each cashflow that takes place during the swap contract. However, the cashflows are based on the value of the floating interest rate which is unknown.
- Imagine the principal is being exchanged at the end of the swap. The cashflows can then be interpreted as the cashflows of a bond.

  The value of a swap position where cashflows based on the floating rate are received and where cashflows based on the fixed rate are paid is therefore equal to the value of a bond whose coupons are based on a floating interest rate minus the value of a bond with fixed coupon payments.

  The swap is thus split into two legs i.e. a fixed rate bond and a floating rate bond. Mathematically, this becomes:

$$V_{swap} = B^{fl.} - B^{Fix.}$$

- Consider a bond that provides coupons based on a floating interest rate. To discount the coupons of this bond, we can use the same floating rate. Therefore, immediately after every coupon payment the bond should be priced at par.

  If the interest rate goes up, the coupon is discounted more heavily, but the coupon gets bigger at the same rate. If the interest rate goes down, the coupon is discounted less heavily, but the coupon gets smaller at the same rate.

  In summary, whatever the movement of the interest rates, the present value of the coupon stays the same. This is because the coupon is determined by the same interest rate that is used to discount the coupon. A floating rate bond is thus priced at par, immediately after every coupon payment, when there still is no accrued interest.

  This can also be shown mathematically by considering the equation for the value of a floating rate bond at its time of inception. At this time, there is no interest accrual.

$$f_{t_0}^{fl.} = \sum_{i=1}^{n} \frac{r_i \cdot N}{(1+r_i)^n} + \frac{N}{(1+r_n)^n}$$

  Note that the first term is a geometric series, so the following applies:

$$S_n = \frac{a \cdot (1+r_n)}{1-r}$$

  Where:

$$\begin{cases} a &= \frac{r_i}{1+r_i} \\ r &= \frac{1}{1+r_i} \end{cases}$$

We can then rewrite the equation for the value of the floating rate bond:

$$\begin{aligned} f_{t_0}^{fl.} &= N \cdot \frac{\frac{r_i}{1+r_i} \cdot (1 + (1 + r_n)^{-n})}{1 - \frac{1}{1+r_i}} \\ &= N \cdot \left[1 - (1 + r_n)^{-n} + \frac{1}{(1 + r_n)^{-n}}\right] \\ &= N \cdot \left[1 - (1 + r_n)^{-n} + (1 + r_n)^{n}\right] \\ &= N \end{aligned}$$

- The formula above tells us that the value of the floating rate bond at its time of inception needs to be equal to the face value of the bond. Note that this is only true if the interest rate curve that is used to determine the coupons is the same as the interest rate curve that is used to discount the coupons back to time t_0.
- This result can easily be confirmed by using an example. Consider for instance a floating rate bond with a principal value of $100 that provides the six-month Libor which is equal to 2%. The time to maturity is one year. Suppose we have just received a coupon payment. The coupon within six months and at the maturity of the bond is given by:

$$\begin{aligned} C &= Pr \cdot c \\ &= \$100 \cdot 0.02 \\ &= \$2 \end{aligned}$$

As we discussed before, the present value of the bond, just after the first coupon payment can be calculated by discounting the cashflows at the prevailing interest rate. The bond is then priced at par. Mathematically:

$$\begin{aligned} P &= \frac{C}{(1 + r_{t_0,t_0+6m})^{6m}} + \frac{C + Pr}{(1 + r_{t_0,t_0+12m})^{12m}} \\ &= \frac{2}{1.02^1} \cdot \frac{102}{1.02^2} \\ &= 100 \\ &= Pr \end{aligned}$$

- We now have all the tools needed to value an interest rate swap. We avoided the problem of not knowing the future cashflows of the floating rate bond by using the coupon rate as the interest rate used for discounting.

- Consider the example where a financial institution has agreed to pay the six-month Libor and receive a fixed interest rate of 8% p.a. s.a. on a principal of \$100.000.000. The swap contract has a remaining time to maturity of 1.25 years. The Libor rates for different maturities are given in the table below. The six-month Libor rate at the time of the last payment was equal to 10.2%. We want to determine the value of the swap.

Time to maturity	Six month Libor
-3 months	10.2%
3 months	10.0%
9 months	10.5%
15 months	11.0%

First, we calculate the present value of the fixed rate bond:

$$\begin{aligned} B_{t_0}^{fix} &= C \cdot e^{r \cdot 3m} + C \cdot e^{r \cdot 9m} + (C + Pr) \cdot e^{r \cdot 512m} \\ &= 4 \cdot e^{0.04 \cdot \frac{3}{12}} + 4 \cdot e^{0.04 \cdot \frac{9}{12}} + 104 \cdot e^{0.04 \cdot \frac{15}{12}} \\ &= \$98.24 \end{aligned}$$

Next, we calculate the present value of the floating rate bond. We only consider the first coupon payment while the bond already has accrued interest at the time of valuation.The other coupons and the principal are discounted at the coupon rate.

The value of the bond will be slightly above the principal value while the bond already has accrued some interest. First we determine the value of the bond at the time of the coupon payment. As we already know, this is the sum of the coupon and the principal value. The value of the bond at the time of valuation t_0 is three months earlier. We therefore need to discount the value of the bond with three months.

$$\begin{aligned} B_{t_0}^{fl.} &= (C_{t_1} + Pr) \cdot e^{t_1 - t_0} \\ &= (\frac{0.102}{2} \cdot 100 \cdot +100) \cdot e^{-0.105 \cdot \frac{3}{12}} \\ &= 105.1 \cdot 0.974091536 \\ &= \$102.38 \end{aligned}$$

Recall that the financial institution is long in the fixed rate bond and short in the floating rate bond. The value of the swap is therefore given by:

$$\begin{aligned} V_{swap} &= B_{t_0}^{fix.} - B_{t_0}^{fl.} \\ &= \$98.24 - \$102.38 \\ &= -\$4.27 \end{aligned}$$

7.2.7 Determining the swap rates

At the time of the inception of the swap contract, the swap rates are determined by a market maker The market maker computes the swap rate in such a way that the swap is of no value to both parties. We know that the value of a long position in fixed rate in the swap contract is given by:

$$V_{swap} = B_{t_0}^{fix.} - B_{t_0}^{fl.}$$

Therefore, it must follow that the value of the floating rate leg is equal to the value of the fixed rate leg. Mathematically, this becomes:

$$B_{t_0}^{fix.} = B_{t_0}^{fl.}$$

We showed that, at the beginning of the contract, the value of the floating leg has to be equal to the principal value of the bond:

$$B_{t_0}^{fl.} = Pr$$

From the two expression above, it follows that the value of the fixed rate leg has to be equal to the principal value. This means that the swap rate, as determined by the market maker has to be equal to the par yield. Mathematically:

$$f_{t_0}^{fix} = N \iff r = \gamma_{par}$$

7.2.8 Example

Consider a five-year swap agreement with semi-annual payments. The five-year swap rate is swapped for the six-month Libor rate. Suppose the six-month, twelve-month and eighteen month Libor rates are equal to 4%, 4.5% and 4.8% with continuous compounding. The fixed swap rate is equal to 5%. We want to determine the two-year Libor rate.

Conceptually, this arrangement corresponds with a two-year bond that provides semi-annual coupons, where the coupon rate is equal to 5% p.a. As we discussed already, this bond should be priced at par. This means that the price of the bond is equal to the principal value of the bond. The price of the bond is calculated as the present value of all future cashflows. Coupons at different points in time are discounted using the corresponding Libor rate. The only unknown variable left in the equation is the Libor rate with a maturity of two years. Mathematically:

$$\begin{aligned} P = &C_{t_1} \cdot e^{-r_{t_0,t_1} \cdot (t_1 - t_0)} + C_{t_2} \cdot e^{-r_{t_0,t_2} \cdot (t_2 - t_0)} + C_{t_3} \cdot e^{-r_{t_0,t_3} \cdot (t_3 - t_0)} \\ &+ (C_{t_4} + Pr) \cdot e^{-r_{t_0,t_4} \cdot (t_4 - t_0)} \end{aligned}$$

$$100 = 2.5 \cdot e^{-0.04 \cdot \frac{6}{12}} + 2.5 \cdot e^{-0.045 \cdot \frac{12}{12}} + 2.5 \cdot e^{-0.048 \cdot \frac{18}{12}} + 102.5 \cdot e^{-r_{t_0,t_4} \cdot \frac{24}{12}}$$

$$r_{t_0,t_4} = 0.04953$$

7.2.9 Overnight index swaps

An overnight indexed swap is a type of interest rate swap. So far, we used the Libor as a proxy of the risk-free interest rate. However, during the financial crisis of 2008, banks became very reluctant to lend to one another. This caused the Libor to sky-rocket. It became apparent that the Libor can change considerably during a short period of time.

An overnight indexed swap rate is a better index for the risk-free rate. An overnight indexed swap is a swap agreement where a fixed rate is exchanged for the geometric average of the overnight interest rate. At the end of each period, the geometric average of the overnight rates is exchanged for the fixed OIS rate.

The geometric average of a set of n elements is n-th root of the product of these elements. When one of the elements is zero, the geometric average is equal to zero. Mathematically:

$$E^{geo.}(i_1, i_2, ..., i_n) = \sqrt[n]{i_1 \cdot i_2 \cdot ... \cdot i_n}$$

Lending money overnight for three months is less risky then lending money for three months. A lot can go wrong in three months time. The risk that something goes wrong overnight is rather small. Furthermore, a swap contract has less risk than a loan because the principal is not exchanged.

The three-month Libor-OIS spread reflects the difference between the credit risk in a three-month loan to bank that is considered to be of acceptable credit quality and the credit risk in continually-refreshed one day loans to banks that are considered to be of acceptable credit quality. This difference is often used as an indicator of stress in the financial system.

Note that an interest rate swap can also be value by replicating the swap with a portfolio of forward rate agreements.

The floating rate leg involves payments that reset periodically. At each reset date, the bond is at par which means that the present value of the bond will be equal to the notional value of the bond.

The coupon rate is determined at each reset date to match the prevailing market rate.

The discounting mechanism also uses the prevailing market rate.

So at each reset date, the value of the floating rate bond will be equal to the notional value of the bond.

swapping risk positions by swapping cash flows.

Describe how swap contracts are similar but differ from a series of forward contracts.

A forward has only a single payment at maturity, a swap typically involves a series of payments in the future.

A single period swap is equivalent to entering a single forward contract. A fixed interest rate and a market reference rate are determined at the inception of the contract. A notional value is determined at the start of the contract this is the contract size of the forward rate agreement. The cash settlement at the maturity of the forward contract is determined by taking the difference between the fixed interest rate and the value of the market reference rate at maturity. This difference is then multiplied by the nominal value of the forward contract.

A one period interest rate swap agreement is equivalent to a forward contract. An interest rate swap can therefore be viewed as a series of forward rate agreements.

However, notice that for a series of forward rate agreements, there will be multiple fixed interest rates. In the case of an interest rate swap, a single fixed rate is determined at applies throughout the whole lifetime of the swap. For an interest rate swap, there is a single fixed interest rate whichcan be interpreted as the multi period breakeven rate at which an investor would be indifferent to:

- paying the fixed swap rate and receiving the respective forward rates or
- receiving the fixed swap rate and paying the resepctive forward rates

In swap contracts, sometimes the notional amount is exchanged at the inception of the contract.

Both are OTC products. So they are not traded on an exchange, they are tailor-made. Both have a symmetric payoff profile Both have counterparty credit risk exposure.

Describe how swap contracts are similar to but different from a series of forward contracts.

Alternative way to price an interest rate swap would be to determine the implied forward rates. The swap rate is the fixed rate that equates the present value of all future expected floating cash flows to the present value of fixed cash flows. The interest payments of the floating rate bond can be determined by multiplying the implied forward rates with the notional value of the bond. These interest payments then need to be discounted back to t_0. This can be done using the zero rates. The value of the floating rate bond then needs to be equated the to value of the fixed rate bond to determine the fixed rate.

The net of fixed and floating differences is exchanged at the end of each period.

7.3 Currency swaps

A currency swap is defined as an exchange of a principal amount in one currency and a principal amount in another currency. There are two parties involved in a currency swap. There are two different currencies each with a corresponding interest rate. The principals are exchanged between the parties at the inception of the swap contract and at the end of the life of the swap contract. The principal amounts swapped at the inception of the contract and at the maturity of the contract need not be the same. However, the exchange rates will need to be determined in advance. Interest payments are made throughout the life of the swap and are based on the interest rate of the currency of the principal received.

7.3.1 Example

We first consider an example where IBM and BP that enter into a currency swap as counterparties. IBM wants to exchange the PS for USD. BP takes the opposite position. The currency swap involves principal amounts of £10.000.000 and $18.000.000. The interest rate on the USD is a fixed interest rate of 6% while the interest rate on the PS is a fixed rate of 5%. The currency swap is thus a fixed-for-fixed currency swap. The length of the contract is equal to five years. The contract has annual settlements. The table below shows the cashflows for IBM in this currency swap. IBM exchanges the principal in USD for a principal in the PS. IBM pays the interest rate that applies to the PS and receives the interest rate that applies to the USD.

Elapsed time	USD cashflow	PS cashflow
0 years	−$18.00	+£10
1 year	+$1.08	−£0.50
2 years	+$1.08	−£0.50
3 years	+$1.08	−£0.50
4 years	+$1.08	−£0.50
5 years	+$19.08	−£10.50

7.3.2 Applications of currency swaps

- Companies use currency swaps to transform assets and liabilities from one currency into another currency. Companies want to transform assets and liabilities because they can hedge currency risks and interest rate risks in doing so.
- Consider the example where a Belgian brewing company sets up a plant in Brazil to supply the local market.
  - To finance this plant, the company enters into a loan with a Belgian bank. The loan and interest payments are therefore in Euro. However, the revenue of the plant is denominated in Brazilian Real. A strong depreciation in the

Brazilian Real would therefore hurt the profits of the company. Naturally, the company wants to hedge away this risk.

  - One solution is to use currency swap agreements. The company would exchange the principal amount of the loan denominated in Euro for an equivalent amount in Brazilian Real. The plant could then be financed in Brazilian Real.
  - Throughout the life of the swaps the company pays interest based on the Brazilian interest rate and receives interest payments based on the Euro interest rate. The interest paid is therefore the difference between the Euro interest received and the Brazilian Real interest paid.
  - Note that the company can use the Euro interest received to pay the interest on the Euro loan it has entered into with the Belgian bank.
  - At the maturity of the swap, the plant has hopefully generated enough revenue to redeem the principal of the swap in Brazilian Real. The company recovers the principal in Euro in exchange for the principal in Brazilian Real.
  - The principal in Euro can then be used to redeem the loan in Euro that is held with the Belgian bank.

- Companies would also want to transform assets and liabilities between different currencies to take advantage of comparative advantages.
- Consider the example of two companies who are able to take out loans in the USD and in the AUD.
  - The interest rates that apply for both companies are given in the table below.

Company name	United States Dollar	Australian Dollar
General Electric	5.0%	7.6%
Qantas Airlines	7.0%	8.0%
Difference	2%	0.4%

  - General Electric has the absolute advantage in both markets. However, Qantas Airlines has a comparative advantage in the market for the AUD. We can therefore set up a swap agreement to the advantage of both parties.

7.3.3 Valuation of currency swaps

To value a currency swap, we have to decompose the swap into two bonds. In this case the two bonds are fixed rate bonds but in different currencies. The value of the swap for the party that receives interests in its domestic currency and pays interests in a foreign currency is given by:

$$V_{swap} = B_{t_0}^{dom.} - S_{t_0} \cdot B_{t_0}^{for.}$$

Where V_{swap} represents the value of the swap, $B_{t_0}^{dom.}$ represents the value of the bond in the domestic currency, $B_{t_0}^{for.}$ represents the value of the bond in the foreign currency and S_{t_0} represents the exchange rate.

7.3.4 Exercise

Suppose the term structure of interest rates is flat in Japan and in the United States. The interest rate that applies for the JPY is equal to 4% p.a. c.c. The interest rate that applies for the USD is equal to 9% p.a. c.c.

A financial institution enters into a currency swap. It exchanges \$10.000.000 for ¥1.200.000.000. Interest payments take place once a year. The financial institution pays interest based on the JPY interest rate and receives interest based on the USD interest rate. The time to maturity of the swap is three years. The current exchange rate for one USD is ¥110.

We want to determine the value of this swap. First, we set up a table with the different cashflows of the swap and their present value. The table is shown below.

Time	Cashflow in USD	Cashflow in JPY	PV cashflow in USD	PV cashflow in JPY
1	\$800.000	¥60.000.000	\$738493, 08	¥57073765, 47
2	\$800.000	¥60.000.000	\$681715, 03	¥54290245, 08
3	\$10.800.000	¥1.260.000.000	\$8495580, 90	¥1084492050, 30
Total			\$9915789, 01	¥1195856060, 85

The value of the USD bond is equal to \$9915789, 01 while the value of the JPY bond is equal to ¥1195856060, 85. The value of the JPY bond, expressed in USD, is given by:

$$\frac{¥1195856060,85}{110\frac{¥}{\$}} = \$10871418,74$$

The value of the swap can therefore be computed in the following way:

$$\begin{aligned} V_{swap} &= B_{t_0}^{dom.} - S_{t_0} \cdot B_{t_0}^{for.} \\ &= \$10871418,74 - \$9915789,01 \\ &= 955629,73 \end{aligned}$$

Technically, there is a difference between an FX swap and a cross currency swap. In the case of a cross-currency swap, both the principal amount and interest on the principal amount is swapped. In the case of an fx-swap, this does not need to be the case.

Interest payments can be based on a floating rates, fixed rates or one party may pay a floating rate while the other party pays a fixed rate.

Note that fx swaps can therefore be used to hedge both foreign exchange rate risks as well as interest rate risks.

Currency swaps can also be used to obtain lower interest rates than they could get if they would borrow directly in a foreign market. Currency swaps don't need to appear on a company's balance sheet, while a loan would.

Another reason why currency swaps are used is to gain access to a foreign currency. Companies may find it difficult to attain the foreign currency through traditional means and may need that foreign currency to make payments.

there is counterparty risk inherent in currency swaps. This means that there is a risk that one of the parties may default on their obligations.

Finally, currency swaps have limited liquidity, which makes it difficult to enter or exit a swap agreement at a favorable rate.

Other financial instruments can be used in lieu of currency swaps. These include forward contracts and call options. Also, instead of using currency swaps, companies can use natural hedges to manage currency risk.

7.4 Other swaps

- An equity swap is an agreement to exchange the total return realized on an equity index for either a fixed or a floating interest rate.
- A credit default swap is a contract that provides insurance against the risk of default of a particular company.
- Swaps are limited only by the imagination of financial engineers and the desire of corporate treasurers and fund managers for exotic structures.

8 Option strategies

In this chapter we will discuss different option trading strategies. An option strategy is defined as a portfolio consisting of options on a given underlying asset, possibly combined with the asset itself. The aim of such a strategy is to create a specific payoff profile. The strategy incorporates a view on the direction and possibly even the volatility of the market of the underlying asset. There are different groups of option strategies:

- The naked option. This portfolio only contains a short position in an option without having a position in the underlying asset. This type of portfolio is used to create an exposure to a certain risk.

  In securities trading in general, a naked position refers to a securities position, long or short, that is not hedged from market risk. Both the potential gain and the potential risk are greater when a position is naked instead of covered or hedged in some way.

- The covered call. A covered call option portfolio contains a short call option and the underlying itself.

  A situation in which an investor writes an option while holding an equal and opposite position on the underlying asset. A covered call option occurs when the investor owns the underlying asset and writes a call so that the underlying is on hand to sell to the option holder if the option is exercised.

- The protective put. A protective put option portfolio contains a long put option and the underlying itself.

  A protective put involves holding a long position in the underlying asset and purchasing a put option with a strike price equal or close to the current price of the underlying asset. A protective put strategy is also known as a synthetic call.

- The option spread. A spread contains multiple options of the same class i.e. only calls or only puts.

  A spread position is entered by buying and selling equal number of options of the same class on the same underlying security but with different strike prices or expiration dates.

- Combinations of the above.

8.1 Option contracts basics

- Before continuing we quickly repeat the basics of options contracts. There are two possible positions in an options contract i.e. the long position and the short position. The long position buys the option contract while the short position sells the option contract. The party that is long posses a right to sell or buy the asset underlying the option contract. The right of the long party entails an obligation for the short party.
- There are different types of option contracts. Option contracts differ in the obligation/right they grant to the party that is short/long. Options that give the long party the right to buy the underlying asset are called call options. Options that give the long party the right to sell the underlying are called put options.
- An option class refers to all the call options or all the put options listed on an exchange for a particular underlying asset.
- An option series refers to a grouping of options on a given underlying security with the same specified strike price and the same expiration month. However, call and put options are parts of separate series. For example, a call option series would include all the available calls on a specific security at a certain strike price that will expire in the same month.
- An option is a non-linear product i.e. the payoff at maturity of the option is not a linear function of the spot price at maturity of the underlying asset. The reason for this is that the long party will not exercise his option until the option is in the money. When the option is out of the money, the payoff at maturity of the option is equal to zero. As long as the option is in the money, the payoff of the option at maturity is a linear function of the spot price of the underlying at maturity. The figure below shows the payoff profile for the different positions in and types of option contracts.

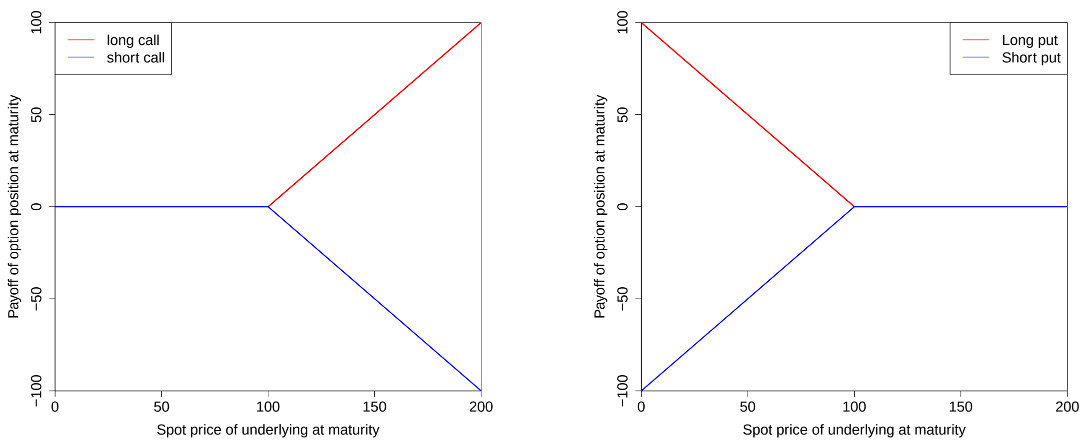


Figure 8.1: Payoff diagram for a call option (left) and a put option (right). The long position is indicated in red while the short position is indicated in blue.

- We now list the payoff functions for all possible combinations of option type and position:

$$f_T^{long\ call} = \begin{cases} 0, & S_T < K \\ S_T - K, & S_T > K \end{cases} \qquad f_T^{long\ put} = \begin{cases} K - S_T, & S_T < K \\ 0, & S_T > K \end{cases}$$

$$f_T^{short\ call} = \begin{cases} 0, & S_T < K \\ K - S_T, & S_T > K \end{cases} \qquad f_T^{short\ put} = \begin{cases} S_T - K, & S_T < K \\ 0, & S_T > K \end{cases}$$

8.2 The covered call

- A covered call option is an option strategy where the portfolio consist of a short call option and the underlying asset of the option contract.
- First, consider the payoff profile of a short call option. It is clear that, if the underlying increases in value, the loss on the short call option will be significant.
- Suppose an investor wants to eliminate the risk of such a loss. One possibility is to buy the underlying asset at the time of entering the short call option. When the underlying increases in value, the investor will not have to pay the higher price to acquire the underlying. This is where the name of the strategy originates.
- We will now derive the payoff function of this portfolio.
  - Suppose the spot price of the underlying at maturity S_T is below the strike price K. The call option will not be exercised. Therefore, the short call position does not provide any cashflows at maturity. However, the investor still posses the underlying a maturity. He can sell the underlying, granting him a positive cashflow of magnitude S_T.
  - Suppose the spot price of the underlying at maturity S_T is greater than the strike price K. The call option will be exercised in that case. The loss in the option strategy is then equal to $K - S_T$. However, this loss is nullified by the long position in the underlying. At maturity, the value of the underlying is equal to the spot price of the underlying at that time S_T.
  - The payoff of the whole strategy is thus given by:

$$f_T^{cov.\ call.} = \begin{cases} S_T, & S_T < K \\ K, & S_T > K \end{cases}$$

  - The payoff function of the covered call can be acquired by simply summing the payoff profiles of the short call option and the payoff profile of the underlying.

- The payoff profile of both the short call option and the covered call is shown in the figure below.

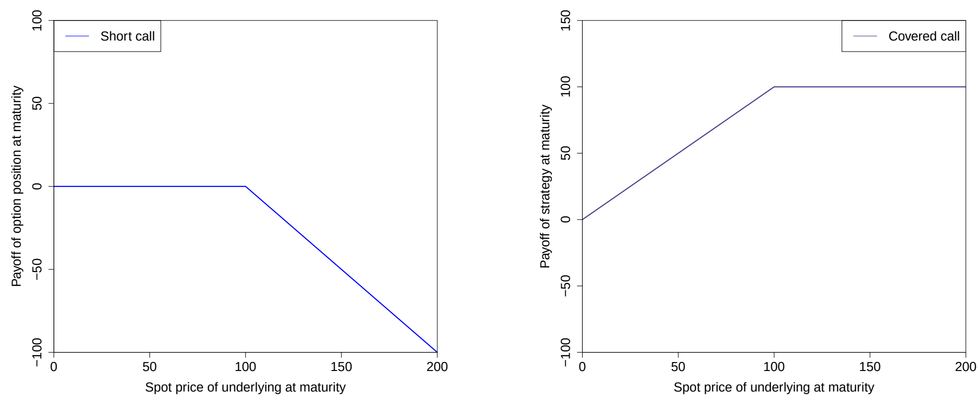


Figure 8.2: Payoff diagram for a short call option (left) and payoff diagram for a covered call option strategy (right).

- In conclusion, a covered call is an option strategy where the portfolio consists of a short call option that is covered by going long spot in the underlying. The main idea is to protect an investor against upwards price movements in the underlying. If the price of the underlying would rise, the underlying is already available. The investor will therefore not have to pay the higher price to acquire the underlying.
- Note that this is not a hedged position. The value of the portfolio is still subject to changes in the value of the underlying. We have a short call position which is hedged to the upside. However, the portfolio can still decrease in value. If the underlying decreases in value, the portfolio will decrease in value. What we did was substitute the risk of a stock increase with the risk of a stock decrease. This time however, potential losses are limited because the downside potential for the stock is limited while the upside potential is unlimited.
- A reverse covered call is an option strategy where the portfolio consists of a long call option and a short position in the underlying asset. We prevent the shorter from having to pay the higher spot price when reacquiring the asset. The payoff profile of this strategy is shown in the figure below.

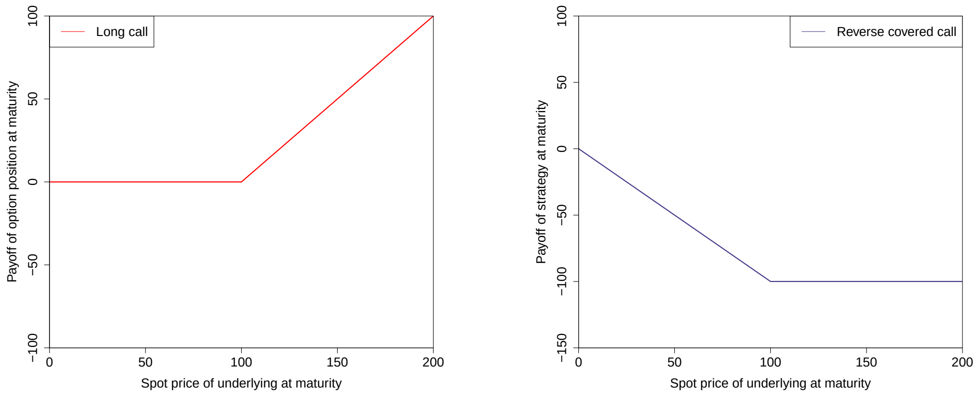


Figure 8.3: Payoff diagram for a long call option (left) and payoff diagram for a reversed covered call option strategy (right).

8.3 The protective put

- A protective put involves holding a long position in the underlying asset and purchasing a put option with a strike price equal or close to the current price of the underlying asset.
- First, consider the payoff profile of a given asset. In the case of a serious market crash, the asset could severely decrease in value. In such a case an investor would want to sell the asset. However, selling the asset might come with high transaction costs. The investor could alternatively choose to add a long put to his portfolio.
- We will now derive the payoff function of this portfolio.
  - Suppose the spot price of the underlying at maturity S_T is above the strike price K. The put option will not be exercised. Therefore, the long put option does not provide any cashflows at maturity. However, the investor still possesses the underlying at maturity which has a value of S_T.
  - Suppose the spot price of the underlying at maturity S_T is smaller than the strike price K. The long put option will be exercised in that case. The payoff of the long put option is equal to $K - S_T$. The price decline in the underlying therefore positively affects the value of the long put options. However this gain is nullified by the decline in the value of the underlying.
  - The payoff of the whole strategy is thus given by:

$$f_T^{prt.\ put.} = \begin{cases} (K - S_T) + S_T = K, & S_T < K \\ S_T, & S_T > K \end{cases}$$

  - The payoff function of the protective put can be acquired by simply summing the payoff profiles of the long put option and the payoff profile of the underlying.

- The payoff profile of both the long put option and the protective put are shown in the figure below.

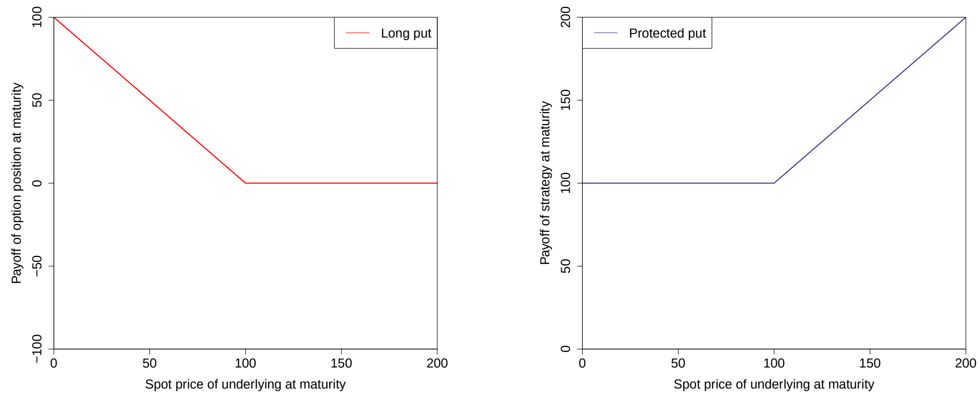


Figure 8.4: Payoff diagram for a long put option (left) and payoff diagram for a protected put option strategy (right).

- In conclusion, a protected put option is an option strategy where the portfolio consists of a long put option and a long position in the underlying value. The long put acts as a form of insurance against declines in the value of underlying asset. Id est the long put ensures that the value of the portfolio does not decline further than a thresholds value. This threshold value is equal to the strike price of the option. Insurance is not a free lunch. It is therefore intuitively clear than an investor will have to pay a premium to acquire the long put option.
- A reverse protective put is an option strategy where the portfolio consists of a short put option and a short position in the underlying asset. We exchange the upside potential of the short for a premium. The payoff diagram of both the short put option and the reverse protective put strategy are shown in the figure below.

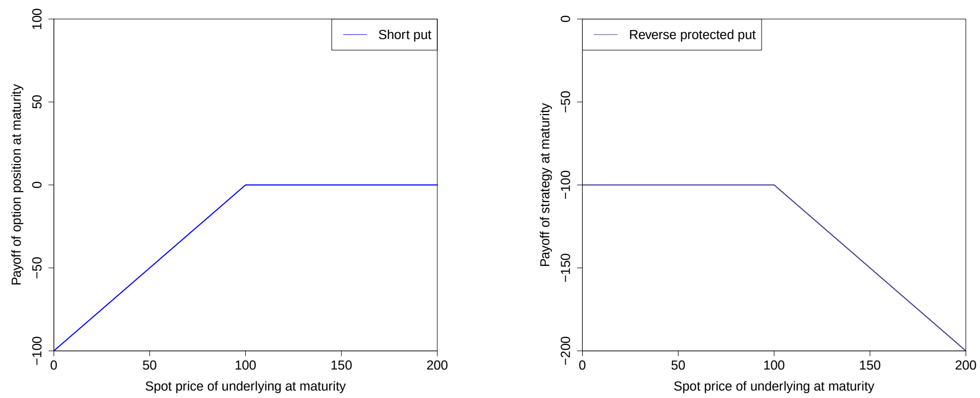


Figure 8.5: Payoff diagram for a short put option (left) and payoff diagram for a reverse protected put option strategy (right).

8.4 Option spreads

- The portfolio of an option spread strategy consists of an equal number of options of the same type on the same underlying but with different maturities or different strike prices. We can discern three major groups of spreads:
  1. Vertical spreads.
  2. Diagonal spreads.
  3. Horizontal spreads which are also known as calendar spreads.
- The naming of the different groups of spreads is purely historical. In the past, option prices were listed in tables where the rows represented different strike prices and the columns represented different maturities. Combining options of different maturities in a portfolio is called a horizontal spread. Combining options of different strike prices in a portfolio is called a vertical spread. Combing options with different maturities and strike prices is called a diagonal spread.
- The figure below shows how options are traditionally listed on an exchange.

	τ_1	τ_2	τ_3	...
K_1				
K_2				
K_3				
...				

- Within the category of spreads reside different strategies that reflect specific outlooks on the market.

8.4.1 Bull-spread with calls

- The portfolio for a bull spread with calls strategy consists of a short call option and a long call option with different strike prices. The strike price of the short is higher than the strike price of the long. Mathematically this becomes:

$$K^{\,short\ call} > K^{\,long\ call}$$

- The payoff of the long call starts to increase when the spot price of the underlying at maturity overtakes the strike price $K^{\,long\ call}$. The payoff of the short call starts to decrease when the spot price of the underlying at maturity overtakes the strike price $K^{\,short\ call}$. The payoff of the portfolio will therefore start to increase when the spot price of the underlying at maturity overtakes the strike price of the long call option. This increase in the payoff will be nullified by the short call option when the spot price of the underlying at maturity overtakes the strike price of the short call option. The figure below illustrates our explanation.

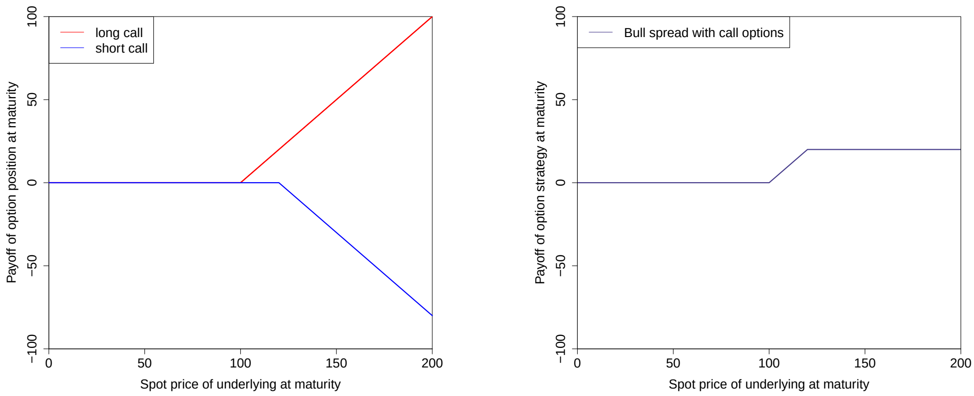


Figure 8.6: Payoff diagram for a long (red) and short (blue) call option (left) and payoff diagram for a bull-spread with calls option strategy (right).

- This option strategy is used by bullish investors who expect a rise in the value of the underlying asset between two boundaries. The lower boundary is equal to the strike price of the long call option while the upper boundary is equal to the strike price of the short call option.
- Our investor expects a rise in the value of the underlying, but not above a certain price. He will therefore first buy the full upside potential of the underlying with the long call option and afterwards sell part of the upside potential to other investors via the short call option.

8.4.2 Bull-spread with puts

- We can also create a bull spread with put options. The portfolio for a bull spread with puts strategy consists of a long put option and a short put option with different strike prices. The strike price of the short is higher than the strike price of the long. Mathematically this becomes:

$$K^{\,short\;put} > K^{\,long\;put}$$

- The payoff of the long put decreases with an increasing spot price of the underlying at maturity and becomes zero at the strike price of the long put. The payoff of the short put increases with an increasing spot price of the underlying at maturity and becomes zero at the strike price of the short put. The payoff of the portfolio will therefore start to increase when the spot price of the underlying at maturity overtakes the strike price of the long put option. This increase in the payoff will halt when the spot price of the underlying at maturity overtakes the strike price of the short put option. The figure below illustrates our explanation.

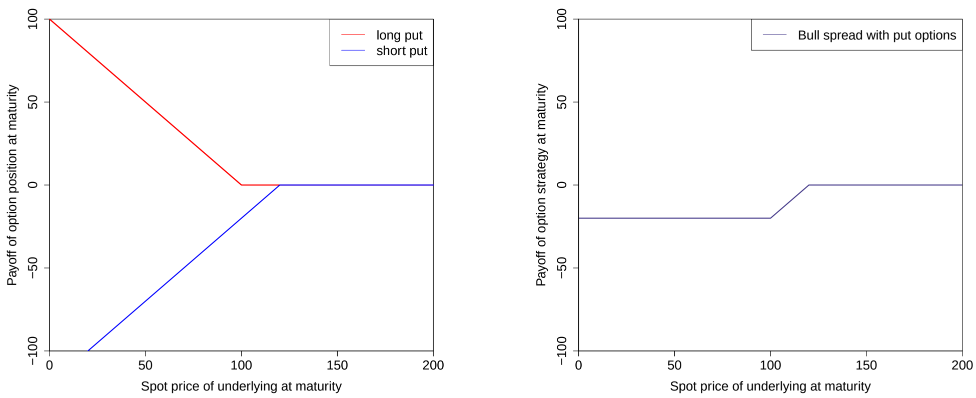


Figure 8.7: Payoff diagram for a long (red) and short (blue) put option (left) and payoff diagram for a bull-spread with puts option strategy (right).

- This option strategy is used by bullish investors who expect a rise in the value of the underlying asset, but want to limit their downside risk.
- Our investor expects a rise in the value of the underlying, and therefore sells put options on the underlying. In doing so he receives an option premium. However, the investor wants to limit the downside potential on the put options he sold. To do this, he will enter long put options with a lower exercise price. The long option compensates for the negative payoff of the short option as long as the spot price at maturity of the underlying is smaller than the exercise price of the long option. The investor thus limits his downside risk while generating income via the option premium.

8.4.3 Bear-spread with calls

- The portfolio for a bear spread with calls strategy consists of a short call option and a long call option with different strike prices. The strike price of the short is lower than the strike price of the long. Mathematically this becomes:

$$K^{\,short\ call} < K^{\,long\ call}$$

- The payoff of the short call starts to decrease when the spot price of the underlying at maturity overtakes its strike price $K^{\,short\ call}$. The payoff of the long call starts to increase when the spot price of the underlying at maturity overtakes its strike price $K^{\,long\ call}$. The payoff of the portfolio will therefore start to decrease when the spot price of the underlying at maturity overtakes the strike price of the short call option. This increase in the payoff will be nullified by the long call option when the spot price of the underlying at maturity overtakes the strike price of the long call option. The figure below illustrates our explanation.

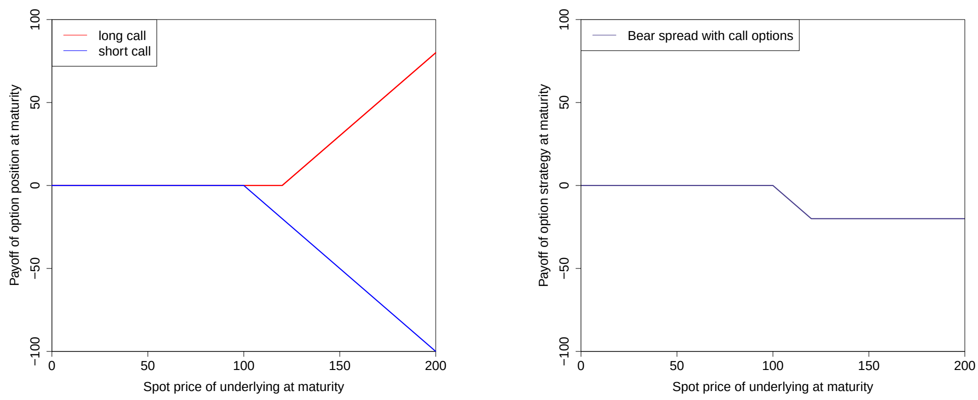


Figure 8.8: Payoff diagram for a long (red) and short (blue) call option (left) and payoff diagram for a bear-spread with calls option strategy (right).

- This option strategy is used by bearish investors who expect a decline in the value of the underlying asset, but want to limit their downside risk.
- Our investor expects a decline in the value of the underlying, and therefore sells call options on the underlying. In doing so he receives an option premium. However, the investor wants to limit the downside potential on the call options he sold. To do this, he will enter long call options with a higher exercise price. The long option nullifies the negative payoff of the short option as long as the spot price at maturity of the underlying is greater than the exercise price of the long option. The investor thus limits his downside risk while generating income via the option premium.

8.4.4 Bear-spread with puts

- The portfolio for a bear spread with puts strategy consists of a short put option and a long put option with different strike prices. The strike price of the short is lower than the strike price of the long. Mathematically this becomes:

$$K^{\,short\ put} < K^{\,long\ put}$$

- The payoff of the short put option increases with the spot price of the underlying at maturity until the spot price of the underlying at maturity is equal to the strike price of the short put option. The payoff of the short put option then becomes zero. The payoff of the long put option decreases with the spot price of the underlying at maturity until the spot price of the underlying at maturity is equal to the strike price of the long put option. The payoff of the long put option then becomes zero. Because the strike price of the short put option is smaller then the strike price of the long put option, the payoff of the portfolio will decrease between both strike prices.

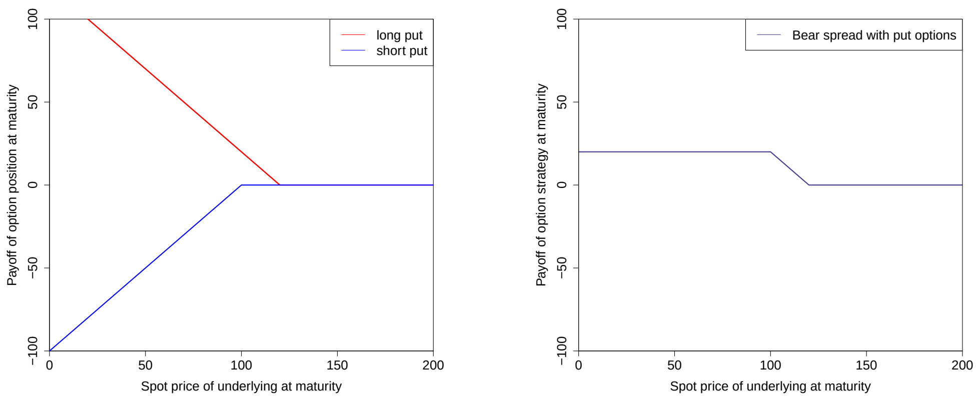


Figure 8.9: Payoff diagram for a long (red) and short (blue) put option (left) and payoff diagram for a bear-spread with puts option strategy (right).

- This option strategy is used by bearish investors who expect a decline in the value of the underlying asset between two boundaries. The lower boundary is equal to the strike price of the short put option while the upper boundary is equal to the strike price of the long put option.
- Our investor expects a decline in the value of the underlying, but not below a certain price. He will therefore first buy the full downside potential of the underlying with the long put option and afterwards sell part of the potential to other investors via the short put option.

8.4.5 Ratio spreads and ratio backspreads

- Ratio spreads and ratio backspreads provide asymmetry in the payoff profile. This is achieved by altering the ratio of the positions in the portfolio. A ratio spread is a strategy where an investor enters into more short positions than long positions. A ratio backspread is a strategy where an investor enters into more long positions than short positions.
- Consider the bull and bear spread option strategies we saw earlier. We can add either a long option or a short option to these portfolios. This does not yield a useful option strategy in all cases. In what follows we show what strategies are useful.
  - Consider the bull spread with calls option strategy. In that case, the strike price of the short was higher than the strike price of the long. A portfolio that incorporates an additional short call option is called a call ratio spread.

$$Portfolio = 1 \cdot C_{K_L}^{long} + 2 \cdot C_{K_H}^{short}$$

  - Consider the bear spread with calls option strategy. In that case, the strike price of the short was lower than the strike price of the long. A portfolio that incorporates an additional long call option is called a call ratio backspread.

$$Portfolio = 2 \cdot C_{K_H}^{long} + 1 \cdot C_{K_L}^{short}$$

  - Consider the bear spread with puts option strategy. In that case, the strike price of the short was lower than the strike price of the long. A portfolio that incorporates an additional short put option is called a put ratio spread.

$$Portfolio = 1 \cdot P_{K_H}^{long} + 2 \cdot P_{K_L}^{short}$$

  - Consider the bull spread with puts option strategy. In that case, the strike price of the short was higher than the strike price of the long. A portfolio that incorporates an additional long put option is called a put ratio backspread.

$$Portfolio = 2 \cdot P_{K_L}^{long} + 1 \cdot P_{K_H}^{short}$$

- The payoff diagrams for all these strategies are shown on the next page.

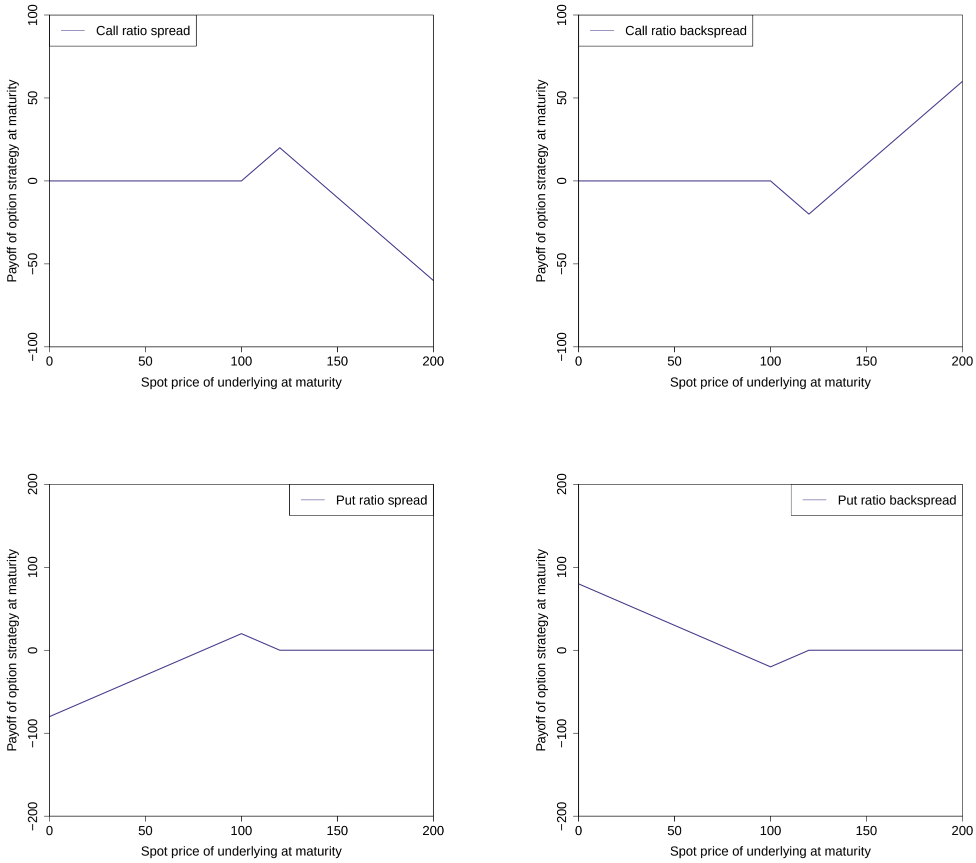


Figure 8.10: From left to right and top to bottom we see the payoff diagrams for a call-ratio spread, a call ratio backspread, a put ratio spread and a put ratio backspread.

8.4.6 Butterfly call spread

- Consider the call ratio spread option strategy. The portfolio consisted of one long call option with a low strike price and two short options with a high strike price. Mathematically:

$$Portfolio = 1 \cdot C_{K_L}^{long} + 2 \cdot C_{K_H}^{short}$$

- The payoff profile possesses some asymmetry in the right tail. An investor might want to remove this tail. This can be done by adding another long call option to the portfolio. We choose the strike price of this long call option in such a way that the the distance between the strike price of the original long and the short options is equal to the distance between the original long and the new long option. Mathematically:

$$K^{\,long\ old} - K^{short} = K^{\,long\ new} - K^{\,long\ old}$$

- The payoff profile of a butterfly call spread is shown in the figure below together with the payoff profiles of the components of the portfolio. Notice that the payoff diagram is always non-negative. This implies that an investor will need to pay a premium to set up a butterfly call spread.

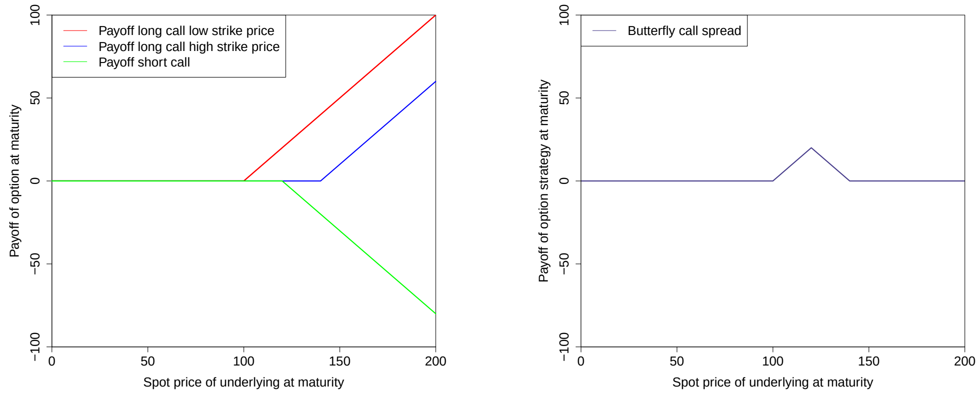


Figure 8.11: Payoff diagrams for a butterfly call spread (right) and its components (left).

8.4.7 Condor spread

- Consider a butterfly call spread where the different options have a different strike price:

$$Portfolio = C_{K_1}^{long} + C_{K_2}^{long} + C_{K_3}^{short} + C_{K_4}^{short}$$

- The strike prices are chosen in such a way that the following statement holds:

$$K_2 - K_1 = K_4 - K_2$$

- The figure on the following page shows the resulting payoff diagram together with the payoff profiles of the components of the portfolio.

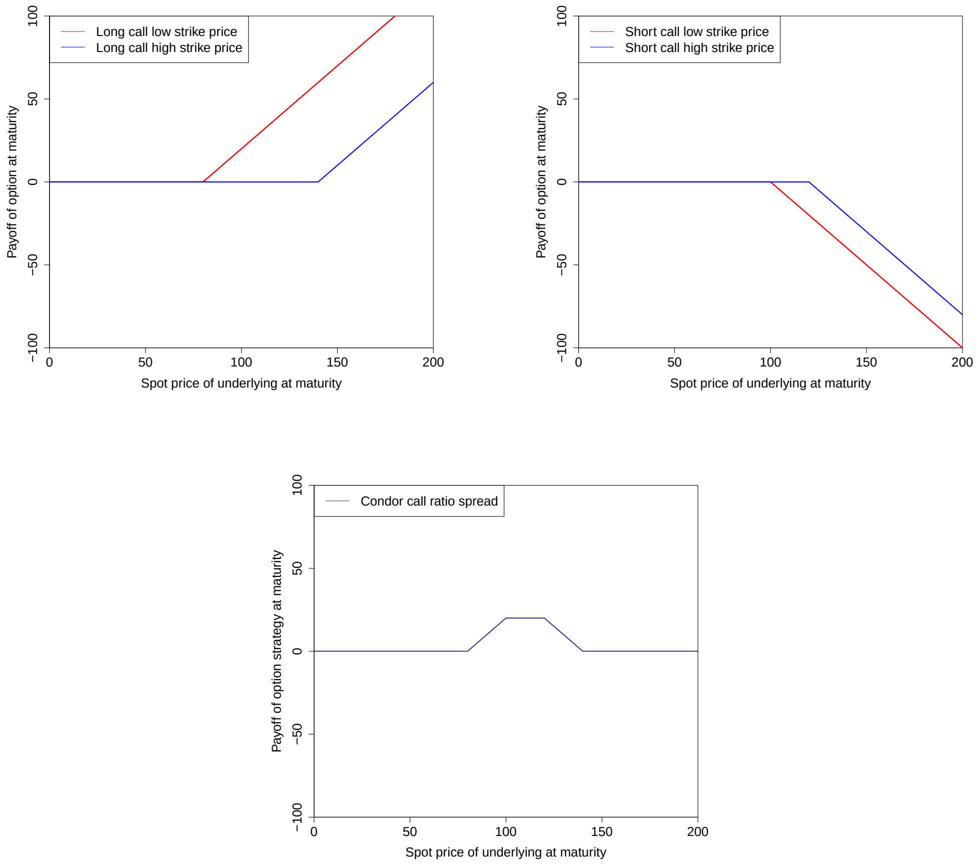


Figure 8.12: Payoff diagrams for a condor spread (bottom) and its components (top).

8.4.8 Box spread

- Consider a portfolio that combines a bull call spread and a bear put spread. This portfolio consists of a long call, a short call, a long put and a short put option:

$$Portfolio = C_{K_1}^{long} + C_{K_2}^{short} + P_{K_3}^{long} + P_{K_4}^{short}$$

- For the strike prices, the following statements hold:

$$K_2 > K_1$$
$$K_3 > K_4$$

- A box spread uses a matching bull call spread and bear put spread. Id est, the lower strike price and higher strike price in both option strategies are equal to one another.

$$K_2 = K_3$$
$$K_1 = K_4$$

- The following table shows the payoff for each component of the box spread option strategy and the payoff of the entire strategy.

Position	Cost	$S_T < K_L$	$K_L < S_T < K_H$	$S_T > K_H$
Long call	$-P(C_L)$	0	$S_T - K_L$	$S_T - K_L$
Short call	$+P(C_H)$	0	0	$K_H - S_T$
Long put	$-P(P_H)$	$K_H - S_T$	$K_H - S_T$	0
Short put	$+P(P_L)$	$S_T - K_L$	0	0
Total cashflow	$P(C_H) + P(P_L)$ $-Pr(C_L) - Pr(P_H)$	$K_H - K_L$	$K_H - K_L$	$K_H - K_L$

- The resulting payoff function is flat. We therefore created a synthetic zero bond. It is thus even possible to create the payoff profile of a bond with an option portfolio.
- An arbitrageur might want to check if the implied interest rate in the box spread is higher than the market interest rate. If this is the case, he could seek to attract funds in the market and set up a synthetic bond using the box spread option strategy.

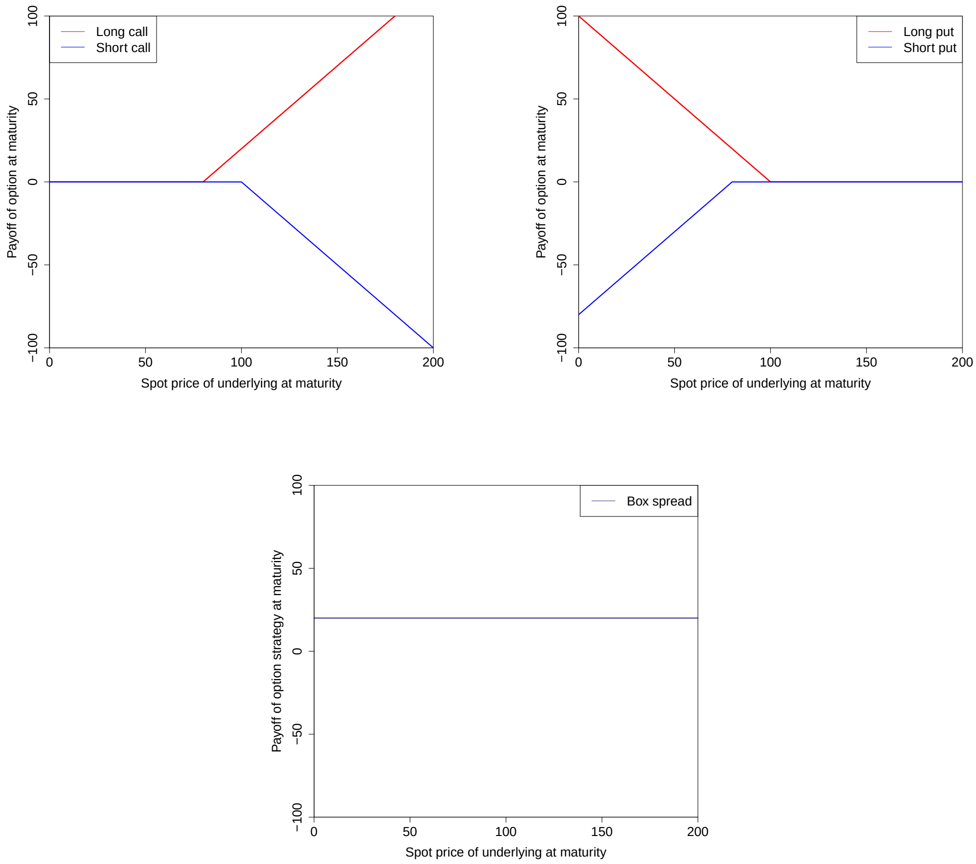


Figure 8.13: Payoff diagrams for a box spread (bottom) and its components (top).

8.5 Combinations

It is also possible to combine options of different classes and even options on different underlying assets. We will only consider combining options of different classes. Here, we discuss four types of option strategies:

1. Straddles. A straddle is an option strategy where the number of put options is equal to the number of call options. The strike price of the put and the call options are the same.
2. Strangles. A strangle is an option strategy where the number of put options is equal to the number of call options. The strike price of the put and the call options are not the same.
3. Strips. A strip is an option strategy where the portfolio consists of two long put options and one long call option. The strike price of the put options and the call option are the same.
4. Straps. A strap is an option strategy where the portfolio consists of two long call options and one long put option. The strike price of the put option and the call options are the same.

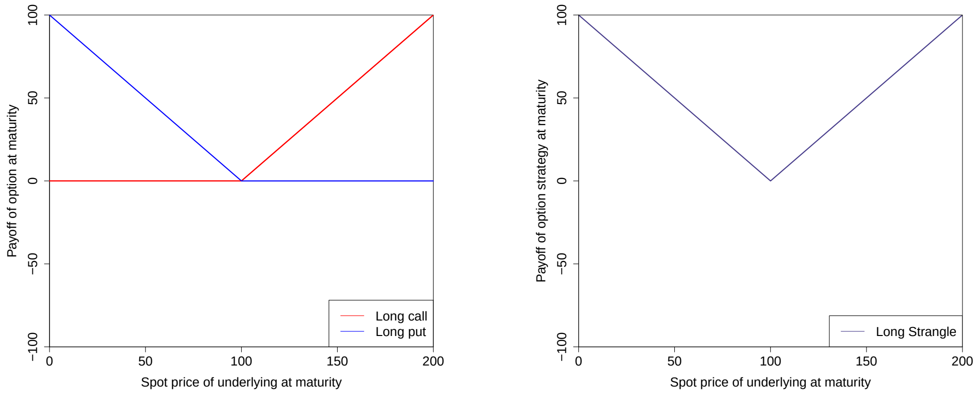


Figure 8.14: Payoff diagrams for a long strangle (left) and its components (right).

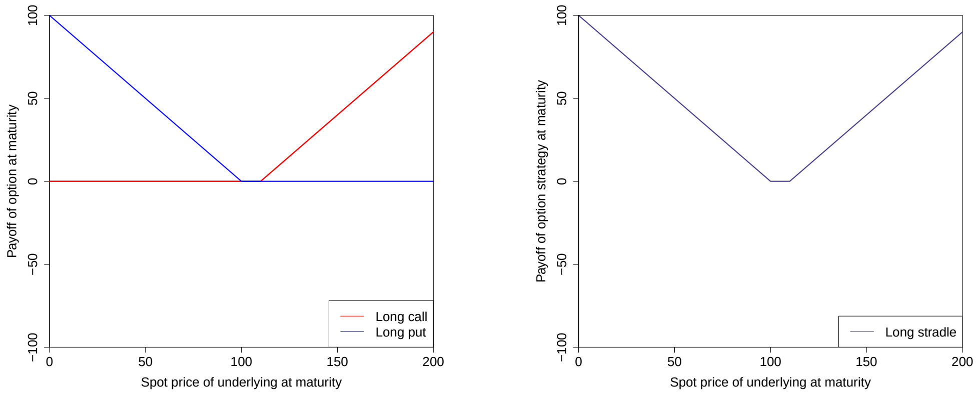


Figure 8.15: Payoff diagram for a long straddle (right) and its components (left).

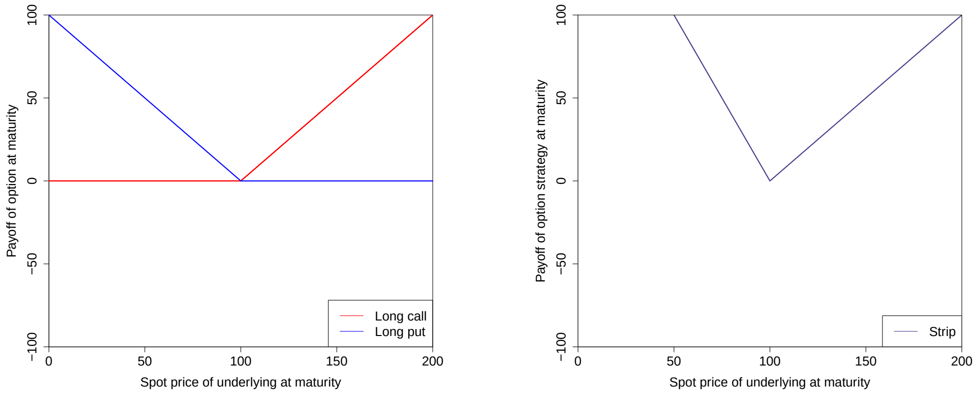


Figure 8.16: Payoff diagrams of a strip (right) and its components (left).

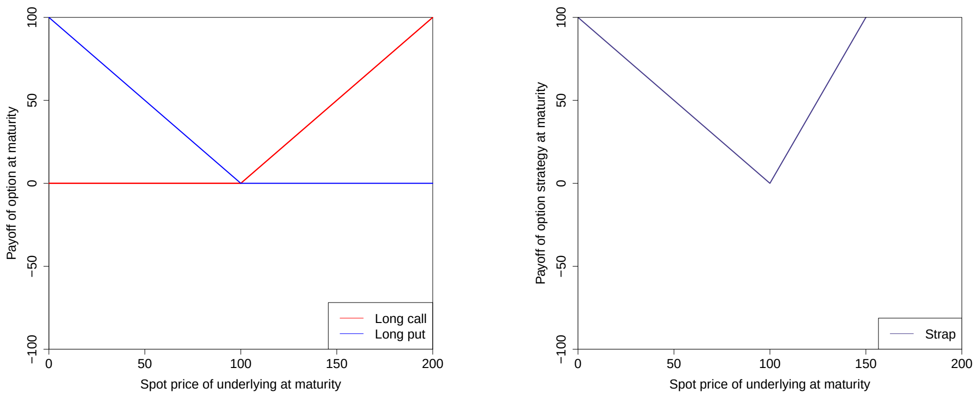


Figure 8.17: Payoff diagrams for a strap (right) and its components (left).

8.6 Reverse engineering

Until now, we combined different options into portfolios. We called these portfolios option strategies. The result is a payoff profile. In practice however, an investor wants to construct a payoff profile using different options. The payoff profile is then given but the composition of the portfolio is unknown. The goal is then to decompose the payoff profile into different options. Id est we want to construct a portfolio for a given payoff profile. We discern four different basic option strategies. These strategies are:

1. The long call.
2. The short call.
3. The long put.
4. The short put.

Options are very unique financial instruments because the slope of the payoff profile changes with the spot price of the underlying at maturity. Each of these strategies have a single kink in their payoff profile. The kink changes the slope of the payoff profile. When the option is out of the money the slope is equal to zero When the option is in the money, the slope is equal to one.

We can thus characterize the four basic option strategies by the sequence of slopes in their payoff profile. For example, the long call option is characterized by the sequence $(0, +1)$ while the short call option is characterized by the sequence $(0, -1)$. The long put option is characterized by the sequence $(-1, 0)$ while the short put is characterized by the sequence $(+1, 0)$. Using these four different basic option strategies, we can construct the payoff profile for any and every possible option strategy. The table below shows the sequence that characterizes each basic option strategy.

Basis strategy	$S_T < K$	$S_T > K$
Long call	0	$+1$
Short call	0	-1
Long put	-1	0
Short put	$+1$	0

8.6.1 The long bull-spread

- First we consider the bull spread option strategy. The payoff profile is given in the figure below and can be characterized by the sequence $(0, +1, 0)$. Id est the slope of the payoff profile is zero at first, then plus one and finally zero again.

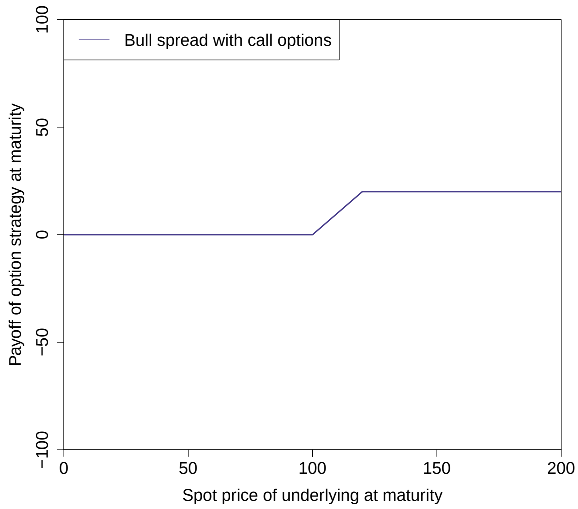


Figure 8.18:

- We can reverse engineer the option portfolio with call options by moving from left to right in the payoff profile.
  - A long call with a strike price of K_L gives the following sequence.

	$S_T < K_L$	$K_L < S_T < K_H$	$S_T > K_H$
Target	0	+1	0
Long call	0	+1	+1
Difference	0	0	+1

  - We add a short call with strike price K_H.

	$S_T < K_L$	$K_L < S_T < K_H$	$S_T > K_H$
Target	0	+1	0
Long call	0	+1	+1
Short call	0	0	−1
Difference	0	0	0

- We will now try to reverse engineer the bull spread using put options. We can do this by moving from right to left in the payoff profile.
  - A short put with a strike price of K_H gives the following sequence.

	$S_T < K_L$	$K_L < S_T < K_H$	$S_T > K_H$
Target	0	+1	0
Short put	+1	+1	0
Difference	+1	0	0

  - Adding a long put with strike price K_L to the portfolio gives us the sequence below.

	$S_T < K_L$	$K_L < S_T < K_H$	$S_T > K_H$
Target	0	+1	0
Short put	+1	+1	0
long put	−1	0	0
Difference	0	0	0

- There is yet another way to construct this option strategy. Note that this strategy is not a spread.
  - First, we use a long forward.

	$S_T < K_L$	$K_L < S_T < K_H$	$S_T > K_H$
Target	0	+1	0
Long forward	+1	+1	+1
Difference	+1	0	+1

  - Next, we add a long put with a strike price of K_L.

	$S_T < K_L$	$K_L < S_T < K_H$	$S_T > K_H$
Target	0	+1	0
Long forward	+1	+1	+1
Long put	−1	0	0
Difference	0	0	+1

– Finally, we add a short call with a strike price of K_H.

	$S_T < K_L$	$K_L < S_T < K_H$	$S_T > K_H$
Target	0	+1	0
Long forward	+1	+1	+1
Long put	−1	0	0
Short call	0	0	−1
Difference	0	0	0

– We can choose the strike prices of the options in such a way that they cancel each other out. The option strategy can then be entered without incurring a cost.

8.6.2 The short butterfly spread

- We now consider the short butterfly spread whose payoff profile is given in the figure below and by characterized by the sequence (0,-1,+1,0).
- We can reverse engineer the option portfolio with, for example call options. As before, we start on the left side of the payoff profile and proceed to the right side of the payoff profile.
  – We start by adding a short call option with strike price K_L.

	$S_T < K_L$	$K_L < S_T < K_M$	$K_M < S_T < K_H$	$S_T > K_H$
Target	0	−1	+1	0
Short call	0	−1	−1	−1
Difference	0	0	+2	+1

  – Next, we use two long call options with strike price K_M.

	$S_T < K_L$	$K_L < S_T < K_M$	$K_M < S_T < K_H$	$S_T > K_H$
Target	0	−1	+1	0
Short call	0	−1	−1	−1
Long call	0	0	+1	+1
Long call	0	0	+1	+1
Difference	0	0	0	+1

– Finally, we add a short call option with strike price K_H.

	$S_T < K_L$	$K_L < S_T < K_M$	$K_M < S_T < K_H$	$S_T > K_H$
Target	0	−1	+1	0
Short call	0	−1	−1	−1
Long call	0	0	+1	+1
Long call	0	0	+1	+1
Short call	0	0	0	−1
Difference	0	0	0	0

8.6.3 Other option strategies

- Some option strategy profiles are difficult to replicate using the method that was used in the previous examples. This is the case when there are discrete jumps in the payoff profile. It is impossible to model these payoff profiles with plain vanilla options.
- To models these strategies, we need exotic options. Binary options are an example of such exotic options. A binary option either provides a fixed cashflow or it does not provide any cashflow. These binary options can be used to model discrete jumps in the payoff profile. The figure below provides the payoff profiles of such binary options.

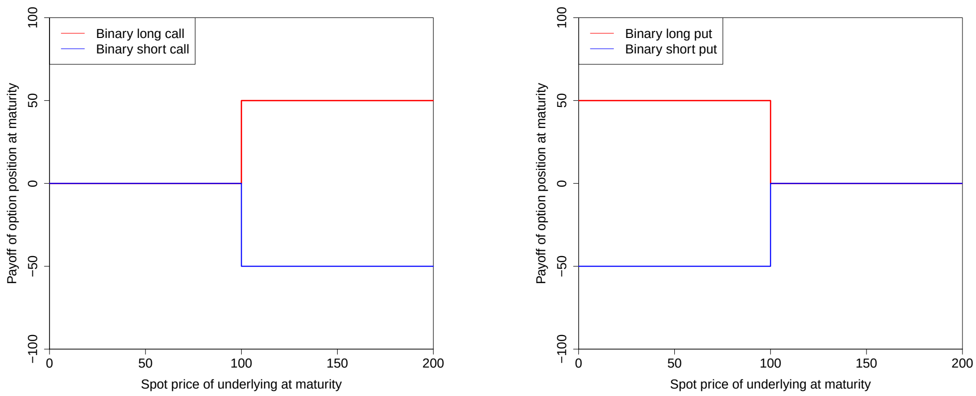


Figure 8.19: Payoff profiles for the binary call options (left) and the binary put options (right).

- Asian options are another example of exotic options. A normal option has a payoff function that compares the strike price with the price of the underlying at maturity. An Asian option compares the average price of the underlying during a period before the maturity with the strike price. An Asian option is therefore less volatile than a normal option.

9 Arbitrage restrictions

The value and thus the price of an options is restricted to some boundaries. In this chapter we will derive these boundaries by looking for the option prices where arbitrage opportunities arise.

9.1 The European call without dividends

- First, we consider the case of the European call option without dividends during the lifetime of the option. We search for restrictions on the option price. The price of an option at a given point in time is naturally the value of the option at that time. Therefore we denote the price of the long call option at time t with:

$$f_t^{long\ call}$$

- We will now go over the different restrictions we can impose on the option price of the long call:

 1. The option price has to be smaller than the price of the underlying. It is clear that no investor is be willing to pay a higher price for the European call option than for the underlying. Mathematically this becomes:

 $$f_t^{long\ call} \leq S_t$$

 2. The option price has to be greater than or equal to zero. If a European call option would have a negative value, an investor could acquire the option and hold onto it until maturity. The investor has the right and thus no obligation to exercise the option. If the option is not in the money at maturity, the investor will let the option expire without any negative consequences. Mathematically this restriction becomes:

 $$f_t^{long\ call} \geq 0$$

 3. The option price has to be greater than or equal to the forward price. Notice that the payoff profile of a long forward contract and a European call option coincide for $S_T > K$. However, the long call option clearly dominates the long forward. This is because the payoff of the long forward can become negative. This means that the value of the long call option has to be at least equal to the value of the long forward.

Mathematically, this restriction becomes:

$$
\begin{aligned}
f_t^{long\ fw.} &\leq f_t^{long\ call} \\
S_t - DP \cdot e^{-r\cdot(T-t)} &\leq f_t^{long\ call} \\
S_t - K \cdot e^{-r\cdot(T-t)} &\leq f_t^{long\ call}
\end{aligned}
$$

If this restriction does not hold, arbitrage opportunities arise. Suppose that the price of the long call option at time t is smaller than the price of the forward contract at the same time:

$$
\begin{aligned}
f_t^{long\ fw.} &> f_t^{long\ call} \\
S_t - K \cdot e^{-r\cdot(T-t)} &> f_t^{long\ call} \\
S_t - K \cdot e^{-r\cdot(T-t)} - f_t^{long\ call} &> 0
\end{aligned}
$$

Consider an investor who, at time t_0, goes short in the underlying asset, uses the proceeds to invest the discounted value of the strike price and to go long in the call option. The equation above tells us that this strategy yields an immediate positive cashflow. I.e. the proceeds from going short in the underlying are greater than the present value of the strike price and the option premium (i.e. the value of the option) combined.

However, we need to make sure that this positive cashflow is not nullified at maturity. We therefore consider the payoff of this portfolio at the maturity of the option contract. The table below displays all possible cashflows of the portfolio at maturity. It is clear that we do not lose the positive cashflow of time t_0 at maturity T.

Arbitrage portfolio	$T : K > S_T$	$T : K < S_T$
Short spot	$-S_T$	$-S_T$
Investment	$+K$	$+K$
Long call option	0	$S_T - K$
Total cashflow	$K - S_T$	0

- In summary, a European call option, in the non-dividend case has:
  - An upper bound. The price of the call option cannot be greater than the spot price of the underlying. Mathematically:

$$f_t^{long\ call} \leq S_t$$

  - A lower bound. The price of the call option cannot be smaller than zero. Mathematically:

$$f_t^{long\ call} \geq 0$$

– A right bound. The price of the call option always has to be greater than the forward price. Mathematically:

$$f_t^{long\ call} \geq f_t^{long\ fw.}$$

- The figure below depicts the payoff of the long forward and the long spot together with the zero line. The area to the right of the long spot payoff, to the left of the long forward payoff and above the zero line is the area to which the long call prices are restricted.

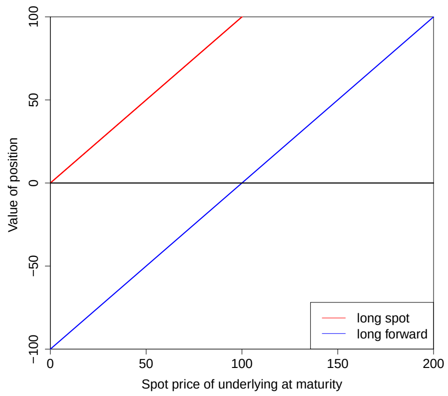


Figure 9.1: Arbitrage restrictions on the European-style long call option in the case of no dividends.

9.2 The European call with discrete dividends

- Next, we consider the case of the European call option, this time with dividends during the lifetime of the option.
- The upper bound and the lower bound are not affected by the dividend. An investor is still not willing to pay a higher price for the option than for the underlying asset. The value of the long option must be greater than zero because the long option grant the investor a right. The investor can therefore choose to let the option expire; in doing so he avoid any possible negative consequences. These restrictions can mathematically be expressed as:

$$f_t^{long\ call} \leq S_t$$
$$f_t^{long\ call} \geq 0$$

- The third restriction on the option price stated that the price of the option should always be greater than or equal to the forward price. However, as we already know, the value of a forward contract is affected by any possible income the underlying asset provides during the lifetime of the forward contract. We corrected the value of the forward by deducting the sum of the present value of each income stream within the lifespan of the forward contract.

- In this case, we apply the same reasoning. The price of the call option then has to be greater than or equal to the value of the forward; where the value of the forward is corrected for the incoming payments during the lifetime of the option contract. Mathematically, this becomes:

$$\begin{aligned} f_t^{long\ call} &\geq S_t - K \cdot e^{-r\cdot(T-t)} - PV(D) \\ &\geq S_t - K \cdot e^{-r\cdot(T-t)} - \sum_{i=1}^{n} D_i \cdot e^{-r\cdot(T-t_i)} \end{aligned}$$

- If this restriction does not apply, arbitrage possibilities arise. Suppose that the value of the option is smaller than the value of the forward, corrected for the incoming payments during the lifetime of the option contract. Mathematically:

$$\begin{aligned} f_t^{long\ call} &\leq S_t - K \cdot e^{-r\cdot(T-t)} - PV(D) \\ 0 &\leq S_t - K \cdot e^{-r\cdot(T-t)} - PV(D) - f_t^{long\ call} \end{aligned}$$

  Consider an investor who, at time t_0, goes short in the underlying asset, uses the proceeds to invest the discounted value of the strike price and the present value of all incoming payments during the lifetime of the option and goes long in the call option. The equation above tells us that this strategy yields an immediate positive cashflow.

- Suppose the underlying provides a single income payment during the lifetime of the long option contract at time t. We can study the cashflows of such a portfolio during its lifetime in the table below. From this it is clear that at no point in time the strategy yields a negative cashflow.

  If the restriction would be violated, an investor could easily set up an arbitrage portfolio that immediately grants him a positive cashflow. There is even a possibility of another positive cashflow at maturity.

Arbitrage portfolio	t_0	t	$T : K > S_T$	$T : K < S_T$
Short spot	$+S_{t_0}$	$-D_t$	$-S_T$	$-S_T$
Investment	$-K \cdot e^{-r\cdot(T-t)}$	0	$+K$	$+K$
Investment	$-PV(D_t)$	$+D_t$	0	0
Long call option	$-f_{t_0}^{long\ call}$	0	$S_T - K$	0
Total cashflow	> 0	0	> 0	0

- In summary, the restrictions from before still apply. In the case where the underlying provides one or more income payments the value of the long call option must be higher than the value of the long forward contract, corrected for the incoming payments during the lifetime of the option contract.

  The incoming payments are discounted and subtracted from the value of the forward contract. The value of the forward contract is therefore decreased. This corresponds to the payoff profile of the long forward being shifted to the right. The area to which the prices of the long call option are restricted is thus enlarged.

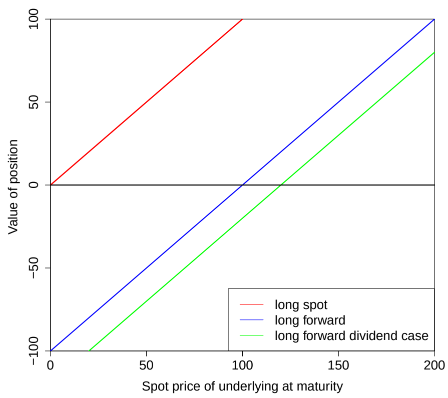


Figure 9.2: Arbitrage restrictions on the European-style long call option, in the case of dividends.

9.3 The American call without dividends

- We now introduce the issue of the time when the option can be exercised. We consider an American long call option. This means the option can be exercised during the entire lifetime of the option.
- The first two restrictions on the value of the option remain valid. The value of the long call option still cannot be greater than the value of the underlying at that point in time. It is also clear that the value of a long call option cannot be less than zero because of the aforementioned reasons. Mathematically, this restrictions were expressed in the following manner:

$$f_t^{long\ call} \leq S_t$$
$$f_t^{long\ call} \geq 0$$

- We take a closer look at the relationship between the European call option and the American call option. The American call option can be exercised at the maturity day, just like the European call option. The American call option can however also be exercised during all the other days, leading up to the maturity of the option. We can easily see that the value of the American call should at least be equal to the value of the European call.

- We conclude that the value of the American call cannot be less than the value of the European call. Mathematically, this becomes:

$$f_t^{Am.\ long\ call} \geq f_t^{Eur.\ long\ call}$$

- The relationship with the value of the forward contract is therefore also straightforward:

$$f_t^{Am.\ long\ call} \geq f_t^{Eur.\ long\ call} \geq S_t - K \cdot e^{-r \cdot (T-t)}$$

- If this restriction does not hold, it is again possible to proof that arbitrage opportunities arise. Suppose that the value of the long American call option is smaller than the value of the long forward. Mathematically:

$$\begin{aligned} f_t^{Am.\ long\ call} &\leq S_t - K \cdot e^{-r \cdot (T-t)} \\ 0 &\leq S_t - K \cdot e^{-r \cdot (T-t)} - f_t^{Am.\ long\ call} \end{aligned}$$

  This corresponds to an investor who, at time t_0, goes short in the underlying asset, uses the proceeds to invest the discounted value of the strike price and to go long in the call option. The equation above tells us that this strategy yields an immediate positive cashflow.

  We can study the cashflows of such a portfolio during its lifetime in the table below. From this it is clear that at no point in time the strategy yields a nett negative cashflow. If the restriction is violated, an investor could easily set up an arbitrage portfolio that immediately grants him a positive cashflow. There is even a possibility of another positive cashflow at maturity.

Arbitrage portfolio	t_0	$T : S_T < K$	$T : S_T > K$
Short spot	$+S_{t_0}$	$-S_T$	$-S_T$
Investment	$-K \cdot e^{-r \cdot (T-t)}$	$+K$	$+K$
Long call option	$-f_{t_0}^{Am.\ long\ call}$	0	$S_T - K$
Total cashflow	> 0	≥ 0	0

- However, we silently assumed that the option was exercised at maturity. The major difference with the European style call option is that the investor can decide to prematurely exercise his option. Whether there is a cashflow at maturity is therefore uncertain. Suppose the investor exercises the option between time t_0 and T. If that were the case, he would receive the intrinsic value of the option. The intrinsic value of the option the difference between the spot price of the underlying at that point in time and the strike price. It is the value that would be received by exercising the option. Mathematically, this becomes:

$$f_t^{Am.\ long\ call} = S_t - K$$

- Consider the restriction with regard to the value of the long call option, which is given by:

$$f_t^{Am.\ long\ call} \geq S_t - K \cdot e^{-r\cdot(T-t)}$$

  It is clear that by exercising the option prematurely, the investor would destroy some value of the option. The investor therefore has no incentive to exercise the option early and will thus wait until maturity.

- All of this implies that an American call option, in the case where there are no dividends provided by the underlying value during the lifetime of the option, will never be exercised early. Exercising the call option early would destroy value of the option. This makes the American style call option equivalent to the European style call option. We can therefore use a pricing model for a European style call options immediately on American style call options. Note that this is only valid in the case of no dividends.

- Graphically, nothing changes with respect to the case of the European call.

9.4 The American call with discrete dividends

- At this point we already know the different restrictions for the value of the long call option. The restrictions in the case of the American call option with discrete dividends follow naturally from the previously stated restrictions. The restrictions for the American-style long call option in the case of dividends are:

  1. The value of the call option cannot be negative.

  $$f_t^{Am.\ long\ call} \geq 0$$

  2. The value of the call option cannot be higher than the value of the underlying.

  $$f_t^{Am.\ long\ call} \leq S_t$$

  3. The value of the American-style call option is greater or equal to the value of the European-style call option.

  $$f_t^{Am.\ long\ call} \geq f_t^{Eur.\ long\ call}$$

  4. The value of the call option is at least equal to the value of the long forward, corrected for the incoming payments during the lifetime of the option contract.

  $$f_t^{Am.\ long\ call} \geq f_t^{long\ forward}$$
  $$f_t^{Am.\ long\ call} \geq S_t - PV(D) - K \cdot e^{-r\cdot(T-t)}$$

- We consider restriction four. It is again possible to proof that arbitrage opportunities arise when this restriction does not hold. Suppose that the value of the long American call option is smaller than the value of the long forward.

Mathematically:

$$f_t^{Am.\ long\ call} \leq S_t - PV(D) - K \cdot e^{-r\cdot(T-t)}$$
$$0 \leq S_t - PV(D) - K \cdot e^{-r\cdot(T-t)} - f_t^{Am.\ long\ call}$$

- An investor could set up an arbitrage portfolio by going short in the underlying. He can use the proceeds to invest the present value of the strike price and the sum of the present value of each future income stream during the lifetime of the option contract and to go long in the call option. This strategy would yield an immediate positive cashflow. We can study the cashflows of this portfolio throughout its lifetime. This shows us that the positive cashflow at time t_0 is not annulled during the lifetime of the portfolio.

Arbitrage portfolio	t_0	t	$T : S_T < K$	$T : S_T > K$
Short spot	$+S_{t_0}$	$-D$	$-S_T$	$-S_T$
Investment	$-K \cdot e^{-r\cdot(T-t)}$	0	$+K$	$+K$
Investment	$-PV(D)$	$+D$	0	0
Long call option	$-f_{t_0}^{Am.\ long\ call}$		0	$S_T - K$
Total cashflow	> 0	0	≥ 0	0

- The arbitrage only works when the investor does not exercise the call option before maturity. However, this does not constitute a problem. We have shown that arbitrage is indeed possible when the restriction is not fulfilled. A situation where the restriction is not met can therefore not prevail in a well functioning market because of arbitrage opportunities.

  Id est, the value of the American-style long call option in the case of dividends, needs to be less than the value of the long forward adapted for the incoming payments. If this is not the case, arbitrage is possible. In other words, the early exercise feature does not influence the restriction.

- Graphically, nothing changes with respect to the case of the European call with discrete dividends. Id est the restriction with regard to the forward contract is less strict. The right boundary for the value of the call option shifts to the right.

- We can ask ourselves whether there are situations where exercising the option prematurely would be justified.

  - As we already know, the value of the long call option is greater than or equal to the value of the long forward. Mathematically:

$$f_t^{Am.\ long\ call} \geq f_t^{long\ forward}$$
$$f_t^{Am.\ long\ call} \geq S_t - K \cdot e^{-r\cdot(T-t)} - PV(D)$$

- When the option is exercised, the payoff is equal to the difference between the spot price of the underlying and the exercise price. This quantity is called the intrinsic value of the call option. Mathematically:

$$f_t^{Am.\ long\ call} = S_t - K$$

- We want to determine in what cases the long call option would be exercised. Id est in what cases the intrinsic value of the call option would exceed the lower boundary for the value of the call option. We rewrite this conditional statement, in terms of the the dividend. Mathematically:

$$\begin{aligned} S_t - K &> S_t - K \cdot e^{-r\cdot(T-t)} - PV(D) \\ PV(D) &> K - K \cdot e^{-r\cdot(T-t)} \end{aligned}$$

- As we can see, the intrinsic value of the long call option exceeds the lower boundary for the value of the long call option when the present value of the dividends is greater than the opportunity cost of exercising the option.
- The opportunity cost is equal to the difference between the strike price and the discounted strike price. This is the interest I that is earned on the cash amount of the strike price K between time t and T. By not exercising the option, the long can hold onto the cash amount of the strike price and earn interest on it. Mathematically:

$$\begin{aligned} TV_t^{long\ call} &= K - K \cdot e^{-r\cdot(T-t)} \\ &= K - PV(K) \\ &= I \end{aligned}$$

- In conclusion, an investor might be tempted to exercise the option early, because it gives him the opportunity to capture the dividend. The option itself does not give the investor access to the dividend.
- The investor compares the time value of the option with the present value of the dividend. If the present value of the dividend is greater than the time value of the option, the investor will exercise the option. If the present value of the of the dividend is less than the time value of the option, the investor will not exercise the option. Mathematically, this becomes:

$$\begin{cases} \sum_{i=1}^{n} D_{t_i} \cdot e^{r\cdot(t_i-t)} < K - K \cdot e^{-r\cdot(T-t)} & \rightarrow \text{exercise} \\ \sum_{i=1}^{n} D_{t_i} \cdot e^{r\cdot(t_i-t)} > K - K \cdot e^{-r\cdot(T-t)} & \rightarrow \text{do not exercise} \end{cases}$$

- Exercising the option will be postponed until the cum dividend day. In doing so the opportunity cost is minimized. If the option would be exercised earlier, the lost time value would be greater.

9.5 The European put option without dividends

- Next, consider the European put option. The first two restrictions are the same as for the long call option. I.e. the value of the long put option cannot be greater than the value of the underlying and the value of the long put option should be greater than zero. Mathematically, this becomes:

$$
\begin{aligned}
f_t^{long\ put} &\geq 0 \\
f_t^{long\ put} &\leq S_t
\end{aligned}
$$

- Furthermore, it is clear that the value of the European put cannot be higher than the discounted value of the strike. This is because the strike price is also the maximal payoff of the long put option at maturity. Mathematically, this becomes:

$$f_t^{long\ put} \leq K \cdot e^{-r \cdot (T-t)}$$

- The value of a European put option must also be greater than or equal to the value of the short forward. If this is not the case, arbitrage opportunities arise. Mathematically, this becomes:

$$
\begin{aligned}
f_t^{long\ put} &\geq f_t^{short\ fw.} \\
&\geq K \cdot e^{-r \cdot (T-t)} - S_t
\end{aligned}
$$

- If the last restriction does not holds, arbitrage opportunities arise. Suppose that the value of the European put is less than the value of the short forward. Mathematically, this corresponds to the following statements:

$$
\begin{aligned}
f_t^{long\ put} &< K \cdot e^{-r \cdot (T-t)} - S_t \\
0 &< K \cdot e^{-r \cdot (T-t)} - S_t - f_t^{long\ put}
\end{aligned}
$$

  In that case an investor could set up an arbitrage portfolio by borrowing the present value of the strike price. He could then use the proceeds to enter a long put option and to go long in the underlying The equation above tells is that this strategy provides an immediate nett positive cashflow. We can study the cashflows of this portfolio throughout its lifetime to see whether the positive cashflow is annulled in the future. This is clearly not the case which means arbitrage is possible if the restriction does not hold true.

Arbitrage portfolio	t_0	$T : S_T < K$	$T : S_T > K$
Long spot	$-S_{t_0}$	$+S_T$	$+S_T$
Investment	$+K \cdot e^{-r \cdot (T-t)}$	$-K$	$-K$
Long put option	$-f_{t_0}^{long\ put}$	$K - S_T$	0
Total cashflow	> 0	0	> 0

- Notice that the lower bound of the long put option is below the intrinsic value of the option. This was not the case for the long call option. From this we derive that the time value for put options can actually be negative. In that case, the value of the put option in the financial markets will be below the intrinsic value. This is because interest can be earned on K. Exercising later would negate this interest partially. When the long put is at its maximum potential, holding out would not make sense, as the option holder would just lose out on interest.

  However, it is important to notice that this is only a possibility. The lower boundary allows for such cases, however in most cases the time value of the option is positive and the intrinsic value of the option will be lower than the value of the option on the financial markets. Mathematically, this becomes:

$$f_t^{long\ put} \geq K \cdot e^{-r\cdot(T-t)} - S_t$$
$$K - S_t > K \cdot e^{r\cdot(T-t)} - S_t$$

- Graphically, the value of the long put option is limited to the area enclosed by the line that represents the value of the underlying and the line that represents the value of the short forward.

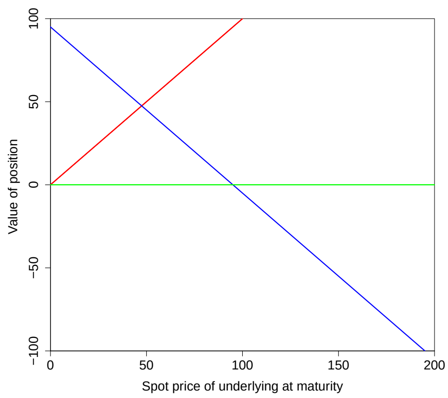


Figure 9.3: Arbitrage restrictions on the European-style long put option, in the case of no dividends.

9.6 The European put with discrete dividends

- We now consider the European put option in the case of discrete income payments. The first three restrictions are the same as for the case without dividends. I.e. the value of the long put option cannot be greater than the value of the underlying, the value of the long put option should be greater than zero and the value of the European put option cannot be higher than the discounted value of the strike. Mathematically, this becomes:

$$f_t^{long\ put} \geq 0$$
$$f_t^{long\ put} \leq S_t$$
$$f_t^{long\ put} \leq K \cdot e^{-r\cdot(T-t)}$$

- The fourth restriction stated that the value of the European put had to be greater than or equal to the value of the short forward. However, in the case of the discrete dividends, the value function of the short forward has to take into account these income payments. The adapted restriction is given by:

$$f_t^{long\ put} \geq K \cdot e^{-r\cdot(T-t)} - S_t + PV(D)$$

- If this restriction does not holds, arbitrage opportunities arise. Suppose that the value of the European put is less than the value of the short forward. Mathematically, this corresponds to the following statements:

$$f_t^{long\ put} < K \cdot e^{-r\cdot(T-t)} - S_t + PV(D)$$
$$0 < K \cdot e^{-r\cdot(T-t)} - S_t + PV(D) - f_t^{long\ put}$$

  An investor can set up an arbitrage portfolio by borrowing the present value of the strike price and the present value of any future dividend payments. The can use the proceeds to go long in the underlying and to go long in a put option. The equation above tells us that these operations provide an immediate nett positive cashflow. To see whether this positive cashflow is annulled in the future, we study the cashflows of the portfolio in the future. It is clear that this is not the case. This means that arbitrage is possible whenever the restriction does not hold true.

Arbitrage portfolio	t_0	t	$T : S_T < K$	$T : S_T > K$
Long spot	$-S_{t_0}$	$+D$	$+S_T$	$+S_T$
Investment	$+K \cdot e^{-r\cdot(T-t)}$	0	$-K$	$-K$
Investment	$+PV(D)$	$-D$	0	0
Long put option	$-f_{t_0}^{long\ put}$	0	$K - S_T$	0
Total cashflow	> 0	0	0	> 0

- Graphically, the value of the long put option is limited to the area enclosed by the line representing the value of the underlying and the line that represents the value of the short forward which is adjusted for the dividend payments during the lifetime of the option. The line representing the value of the short forward is shifted to the right. This means that the restriction on the long put option is less strict in the case of dividend payments.

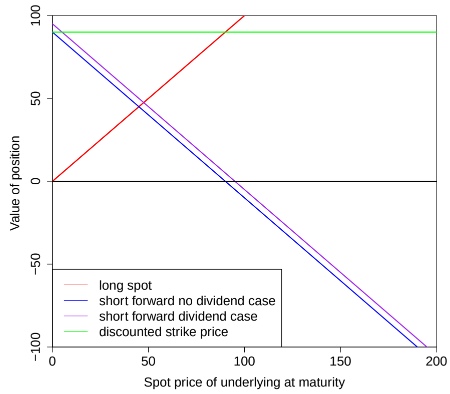


Figure 9.4: Arbitrage restrictions on the European-style long put option, in the case of dividends.

9.7 The American put option without dividends

- Now we discus the case of the American style long put option. The first three restrictions are the same as for the European long put option. I.e. the value of the long put option cannot be greater than the value of the underlying and the value of the long put option should be greater than zero and the value of the long put should not be greater than the present value of the strike price. Mathematically, this becomes:

$$f_t^{Am.\ long\ put} \geq 0$$
$$f_t^{Am.\ long\ put} \leq S_t$$
$$f_t^{Am.\ long\ put} \leq K \cdot e^{-r\cdot(T-t)}$$

- The fourth restriction on the value of the long American put option is somewhat different from the European case. It is clear that the value of the option cannot be lower than the intrinsic value of the option because the option can be exercised at any point in time. If the value of the American put option is lower than its intrinsic value, the option would be exercised immediately. Mathematically, this restriction becomes:

$$f_t^{Am.\ long\ put} \geq K - S_t$$

- As we already discussed, the value of the long put option has to greater than or equal to the value of the short forward. However, we can easily see that the previous restriction is less strict than this restriction. This restriction is therefore of no importance for the long American put option. Mathematically:

$$f_t^{Am.\ long\ put} \geq K \cdot e^{-r\cdot(T-t)} - S_t$$
$$K - S_t > K \cdot e^{-r\cdot(T-t)} - S_t$$

- As the lower boundary for the value of the long put option in the market is equal to the intrinsic value of the long put in the market, early exercise will not be useful. The time value of the option has to be greater or equal to zero. The exception to this rule is an option where the remaining upside potential is zero or almost zero. The time value of the option would then be equal to the interest rate that is earned on the proceeds of the payoff of the option. Holding onto the option would then be of no further benefit.
- Graphically, the value of the American-style long put is limited to the area enclosed by the lines representing the intrinsic value of the long put option and the line representing the spot price of the underlying.

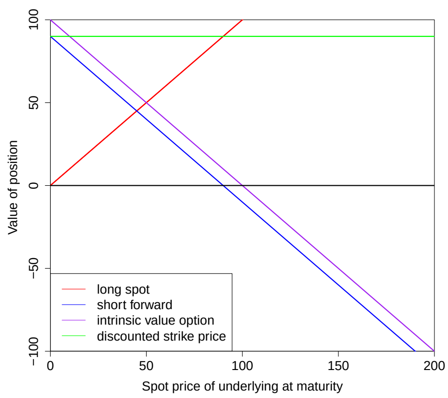


Figure 9.5: Arbitrage restrictions on the American-style long put option, in the case of no dividends.

9.8 The American put option with dividends

- We now consider the case of the American put option on an underlying that provides dividends during the lifetime of the option. The restrictions are the same as for the case of the American-style long put option without dividends. Id est, the value of the long put cannot be greater than the value of the underlying, the value of the long put cannot be greater than the present value of the strike price, the value of the long put should be greater than zero and the value of the long put option cannot be greater than the value of the short forward, adjusted for the dividend payments. The value of the long put also cannot be less than its intrinsic value. Mathematically, this becomes:

$$\begin{aligned} f_t^{long\ put} &\leq S_t \\ f_t^{long\ put} &\leq K \cdot e^{-r\cdot(T-t)} \\ f_t^{long\ put} &\geq 0 \\ f_t^{long\ put} &\geq PV(K) - S_t + PV(D) \\ \left(f_t^{long\ put}\right. &\geq \left. K - S_t\right) \end{aligned}$$

- We are interested in the case where the intrinsic value of the option is greater than the value of the short forward, adjusted for the dividend payments. This shows us when it would be beneficial to exercise the option prematurely. Mathematically:

$$\begin{aligned} f_t^{Long\ put} &\geq K \cdot e^{-r\cdot\tau} - S_t + PV(D) + \left[K - K \cdot e^{-r\cdot\tau}\right] \\ &\geq K - S_t + PV(D) \\ K - S_t + PV(D) &< K - S_t \\ PV(D) &< 0 \end{aligned}$$

- Graphically, the value of the American-style put option is restricted to the area enclosed by the line representing value of the underlying and either the line representing the value of the short forward, or the ling representing the intrinsic value of the option.

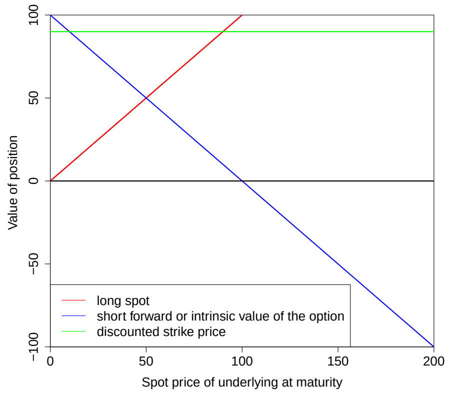


Figure 9.6: Arbitrage restrictions on the American-style long put option, in the case of dividends.

10 Put call parity

10.1 The European put call parity

- We first take a look at the put-call parity in the case of European-style options. From the graph below, it is easy to see that the following relation must hold:

$$\begin{aligned} f_t^{long\ call} - f_t^{long\ put} &= f_t^{long\ forward} \\ &= S_t - K \cdot e^{-r\cdot(T-t)} \\ f_t^{long\ call} + K \cdot e^{-r\cdot(T-t)} &= S_t + f_t^{long\ put} \end{aligned}$$

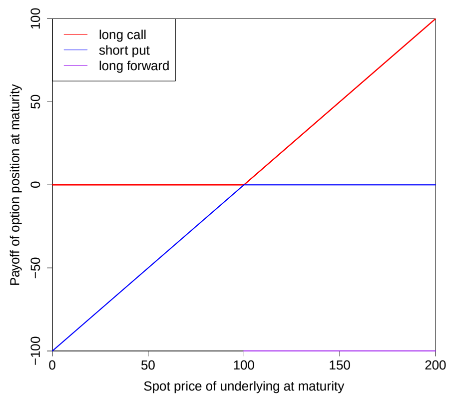


Figure 10.1: The payoff profile of a long call option, a short put option and the corresponding long forward contract.

- The proof of this relationship is again based on the no-arbitrage assumption. If the relationship does not hold, it is possible to set up aa arbitrage strategy.
  - Suppose first that the following holds true:

$$\begin{aligned} f_t^{long\ call} + K \cdot e^{-r\cdot(T-t)} &\geq S_t + f_t^{long\ put} \\ S_t + f_t^{long\ put} - K \cdot e^{-r\cdot(T-t)} - f_t^{long\ call} &\geq 0 \end{aligned}$$

– It is then possible to set up strategy that provides an immediate net positive cashflow. This portfolio consists of a short put option, a long call option. a short spot position and an investment with a principal amount of the present value of the strike price. We now look at the cashflows of this portfolio throughout its lifetime in more detail. To ensure that the cashflow at time t_0 is not annulled in the future. This is clearly not the case.

Arbitrage portfolio	t_0	$T : S_T < K$	$T : S_T > K$
Short spot	$+S_{t_0}$	$-S_T$	$-S_T$
Investment	$-K \cdot e^{-r\cdot(T-t_0)}$	$+K$	$+K$
Long call option	$-f_{t_0}^{long\ call}$	0	$S_T - K$
Short put option	$+f_{t_0}^{short\ put}$	$S_T - K$	0
Total cashflow	> 0	0	0

– Suppose now that the following holds true:

$$f_t^{long\ call} + K \cdot e^{-r\cdot(T-t)} \leq S_t + f_t^{long\ put}$$
$$0 \leq f_t^{long\ call} + K \cdot e^{-r\cdot(T-t)} - S_t - f_t^{long\ put}$$

– It is again possible to set up an arbitrage portfolio that provides an immediate positive net cashflow. This time, the portfolio consists of a short call option, along put option, a long spot position and a loan with a principal amount equal tot the present value of the strike price. We can again take a more detailed look at the cashflows of this strategy throughout its lifetime to make sure that the positive cashflow at time t_0 is not annulled in the future. This is clearly not the case.

Arbitrage portfolio	t_0	$T : S_T < K$	$T : S_T > K$
Long spot	$-S_{t_0}$	S_T	S_T
Loan	$K \cdot e^{-r\cdot(T-t_0)}$	$-K$	$-K$
Short call option	$f_{t_0}^{long\ call}$	0	$K - S_T$
Long put option	$-f_{t_0}^{short\ put}$	$K - S_T$	0
Total cashflow	> 0	0	0

- We conclude that the put call parity must hold, otherwise arbitrage opportunities arise. Arbitrage opportunities can exist, but cannot persist.

- We can generalize the European put call parity for the case of dividends. In that case, the value of the long forward has to take into account the income streams of the underlying asset. We can easily derive this more general relationship. If this relationship does not hold, arbitrage opportunities arise. The proof for this is very similar to the ones above.

$$f_t^{long\ call} - f_t^{long\ put} = f_t^{long\ forward}$$
$$f_t^{long\ call} - f_t^{long\ put} = S_T - K \cdot e^{-r\cdot(T-t)} - PV(D)$$

10.2 Applications of the put call parity

- The put call parity can be useful in the following cases:
  1. If we can find a formula to price call options, the put call parity can be used to price corresponding put options.
  2. The put call parity can be used to search for arbitrage opportunities.
  3. The put call parity can be used to create synthetic instruments.
  4. The put call parity can be used to predict how price changes in the underlying affect the difference between the call and the put price.
  5. It is possible to back out the dividend estimate of the market.

10.3 The American put call parity no dividend case

- In the previous section we derived the expression for the European put call parity:

$$f_t^{Eur.\ long\ call} + K \cdot e^{-r\cdot(T-t)} = S_t + f_t^{Eur.\ long\ put}$$

- In the case where the underlying provides no dividends, the following statements hold true:

$$f_t^{Eur.\ long\ call} = f_t^{Am.\ long\ call}$$
$$f_t^{Eur.\ long\ put} \leq f_t^{Am.\ long\ put}$$

  We can therefore rewrite the put call parity:

$$f_t^{Eur.\ long\ call} - f_t^{Eur.\ long\ put} = S_T - K \cdot e^{-r\cdot(T-t)}$$
$$f_t^{Am.\ long\ call} - f_t^{Eur.\ long\ put} = S_T - K \cdot e^{-r\cdot(T-t)}$$
$$f_t^{Eur.\ long\ put} = f_t^{Am.\ long\ call} - S_T + K \cdot e^{-r\cdot(T-t)}$$
$$f_t^{Am.\ long\ put} \geq -S_T + K \cdot e^{-r\cdot(T-t)} + f_t^{Am.\ long\ call}$$

- There also exists a lower bound for the difference between the value of the American style long call and the long put. This lower bound is given by the equation below.

$$f_t^{Am.\ long\ call} - f_t^{Am.\ long\ put} \geq S_t - K$$

We can proof that this relationship is valid by proving that contradicting statements cannot hold true because they give rise to arbitrage opportunities. If this condition does not hold up, the following must hold true:

$$\begin{aligned} f_t^{Am.\ long\ call} - f_t^{Am.\ long\ put} &\leq S_t - K \\ 0 &\leq S_t - K - f_t^{Am.\ long\ call} + f_t^{Am.\ long\ put} \end{aligned}$$

We can thus construct a strategy that provides an immediate positive cashflow. The portfolio consists of a short position in the underlying, an investment with a principal amount equal to the strike price, a long position in a call option and a short position in a put option. We take a more detailed look at the cashflows the strategy provides throughout its lifetime in the table below.

Arbitrage portfolio	t_0	$T: S_T < K$	$T: S_T > K$
Short spot	S_{t_0}	$-S_T$	$-S_T$
Investment	$-K$	$+K \cdot e^{r\cdot(T-t_0)}$	$+K \cdot e^{r\cdot(T-t_0)}$
Long call option	$-f_{t_0}^{long\ call}$	0	$S_T - K$
Short put option	$+f_{t_0}^{short\ put}$	$S_T - K$	0
Total cashflow	> 0	> 0	> 0

It is clear that the strategy not only provides a net positive cashflow at time t_0, but also at maturity T of the option contracts. There are no net negative cashflows during the lifetime of the portfolio.

However, there is still a loose end. What happens if the short American style put is being exercised before maturity, by the counterparty that is long? We can say with certainty that the put option will only be exercised when the strike price is greater than the spot price.

Suppose we close out all other positions in the portfolio, at the same time. This would mean buying the underlying spot, closing out the investment prematurely and selling the long call option. The net cashflow would then be equal to:

$$\begin{aligned} f_t^{Port.} &= -S_t + K \cdot e^{r\cdot(t-t_0)} + f_t^{Am.\ long\ call} - K + S_t \\ &= K \cdot e^{r\cdot(t-t_0)} - K + f_t^{Am.\ long\ call} \\ &\geq 0 \end{aligned}$$

We can see that the portfolio still provides a nett positive cashflow which means that the restriction must hold, otherwise arbitrage opportunities arise.

10.4 The American put call parity dividend case

- Suppose the underlying provides income streams during the life of the option contracts. In that case, the upper bound of the put call parity changes. The put-call parity then becomes:

$$S_t - PV(D) - K \leq f_t^{Am.\ long\ call} - f_t^{Am.\ long\ put} \leq S_t - K \cdot e^{-r\cdot(T-t)}$$

- We can again proof that this relationship holds true using the no-arbitrage principle. First, we take a look at the upper bound restriction. If the upper bound restriction does not hold, the following statements have to be true:

$$f_t^{Am.\ long\ call} - f_t^{Am.\ long\ put} > S_t - K \cdot e^{-r\cdot(T-t)}$$
$$f_t^{Am.\ long\ call} - f_t^{Am.\ long\ put} - S_t + K \cdot e^{-r\cdot(T-t)} > 0$$

We can then set up an arbitrage portfolio. The portfolio consists of a short position in an American style call option, a long position in an American style put option, a long position in the underlying asset and a loan where the principal amount is equal to the present value of the strike price. This strategy yields an immediate nett positive cashflow. The table below outlines the different cashflows during the lifetime of the strategy. We can clearly see that there are no nett negative cashflows at any point in time during the lifetime of the portfolio.

Arbitrage portfolio	t_0	$T : S_T < K$	$T : S_T > K$
Long spot	$-S_{t_0}$	$+S_T$	$+S_T$
Loan	$+K \cdot e^{-r\cdot(T-t)}$	$-K$	$-K$
Short call option	$+f_{t_0}^{long\ call}$	0	$K - S_T$
Long put option	$-f_{t_0}^{long\ put}$	$K - S_T$	0
Total cashflow	> 0	0	0

- Consider the situation where the holder of the call option exercises the option prematurely. If premature exercise is beneficial, the party that is long will hold out on exercising the option until the cum-dividend day. The option will only be exercised when the spot price of the underlying at that time is greater than the strike price. Mathematically:

$$S_t > K$$

Suppose that we close out all other positions in the portfolio at the same time. This yields a positive cashflow from the investment, a negative cashflow from the short call option and a positive cashflow from selling the long put option.

We can easily see that the magnitude of the cashflow that results from selling the underlying spot is greater than the magnitude of the cashflows that results from the short call option and the loan. Mathematically, this becomes:

$$\begin{aligned} {f_t}^{Port.} &= K - S_t + {f_t}^{Am.\ long\ put} + S_t - K \cdot e^{-r\cdot(T-t)} \\ &= K - K \cdot e^{-r\cdot(T-t)} + {f_t}^{Am.\ long\ put} \\ &> 0 \end{aligned}$$

- Next, we take a look at the lower bound restriction. The lower bound of the put-call parity for American-style options where the underlying provides an income stream, is given by:

$$S_t - PV(D) - K \leq {f_t}^{Am.\ long\ call} - {f_t}^{Am.\ long\ put}$$

- We can proof that this relationship holds true using the no-arbitrage principle. If the lower bound restriction does not hold, the following statements have to be true:

$$S_t - PV(D) - K > {f_t}^{Am.\ long\ call} - {f_t}^{Am.\ long\ put}$$
$$S_t - PV(D) - K - {f_t}^{Am.\ long\ call} + {f_t}^{Am.\ long\ put} > 0$$

We can set up an arbitrage strategy. The portfolio consists of a long position in an American style call option, a short position in an American style put option, a short position in the underlying asset, an investment where the principal amount is equal to the present value of the strike price and an investment where the principal amount is equal to the sum of the present value of each future income stream that takes place during the lifetime of the options. The table below outlines the different cashflows during the lifetime of the strategy.

Arbitrage portfolio	t_0	t	$T : S_T < K$	$T : S_T > K$
Short spot	$+S_{t_0}$	$-D$	$-S_T$	$-S_T$
Investment	$-K$	0	$+K \cdot e^{r\cdot(T-t)}$	$+K \cdot e^{r\cdot(T-t)}$
Investment	$-PV(D)$	$+D$	0	0
Long call option	$-f_{t_0}^{long\ call}$	0	0	$S_T - K$
Short put option	$+f_{t_0}^{long\ put}$	0	$S_T - K$	0
Total cashflow	> 0	0	> 0	> 0

We can clearly see that the portfolio only provides nett positive cashflows during the lifetime of the portfolio. We conclude that the lower boundary of the put-call parity for American-style options must hold, also in the case of dividends, because arbitrage opportunities cannot persist.

- Consider however the situation where the short American-style put option is being exercised by the long prematurely. As we know, the long will only exercise the option when the strike price is higher than the actual spot price of the underlying:

$$K > S_t$$

Suppose we close out all other positions, at the same time. In that case, the American-style call option is out of the money, but it will still have a positive value due to the time value of the option. We are then left with the positive cashflows from the investments and the negative cashflows from the short position in the underlying and the short American put. We can easily see that the magnitude of the cashflow that results from investments is greater than the magnitude of the cashflows that results from the short put option and the short position in the underlying. This means that the total cashflow has to be positive. Mathematically, this becomes:

$$\begin{aligned} f_t^{portfolio} &= -S_t + PV(D) \cdot e^{r \cdot (t-t_0)} + K \cdot e^{r(t-T_0)} + f_t^{Am.\ long\ call} + S_t - K \\ &= PV(D) \cdot e^{r \cdot (t-t_0)} + K \cdot e^{r(t-T_0)} - K + f_t^{Am.\ long\ call} \\ &> 0 \end{aligned}$$

11 Determinants of option prices

- There are many factors that have an impact on the value of a position in an option contract. We will discuss each of these factors in this chapter. The main determinants of option prices are:
  - S_t: the value of the underlying.
  - K: the height of the strike price.
  - τ: the time to maturity.
  - r: the riskless interest rate.
  - σ: the volatility.
- First, consider the current value of the underlying S_t. When a long call option is exercised, the payoff of the option is equal to the amount by which the price of the underlying exceeds the exercise price. The long call option therefore becomes more valuable as the price of the underlying increases. When the long put option is exercised, the payoff is equal to the the amount by which the exercise price exceeds the current value of the underlying. The long put option therefore becomes less valuable as the current value of the underlying increases.

$$f_t^{long\ call} = S_t - PV(K)$$
$$f_t^{long\ put} = PV(K) - S_t$$

- Next, consider the influence of the height of the strike price K. As mentioned before, the payoff of the long call option when it is exercised is equal to the amount by which the price of the underlying exceeds the exercise price. The long call option therefore becomes less valuable as the strike price increases. The payoff of the long put option is equal to the amount by which the exercise price exceeds the price of the underlying. The long put option therefore becomes more valuable as the strike price increases.

$$f_t^{long\ call} = S_t - PV(K)$$
$$f_t^{long\ put} = PV(K) - S_t$$

- We now discuss the influence of the time to maturity τ on the value of options.
  - Consider the case of American-style options. As the time to maturity decreases, the time value of the option decreases. The long call and the long put option therefore become less valuable as the time to maturity decreases. However, these statements do not take into account any possible dividends the underlying provides during the lifetime of the option contract.
  - Consider the case of European-style options. As we discussed before, the long call option always has a positive time value. This means that the long call option will become less valuable as the time to maturity decreases.

    The time value of the long put option however, can be negative. This will be the case when the long put is at or close to its maximal payoff value. Under those circumstances, holding out will not be beneficial. This means that we cannot make any statements about the relationship between the European-style long put option and the remaining time to maturity.
- Consider the influence of the interest rate r on the value of options. When the riskless interest rate increases, the present value of the exercise price decreases. The payoff of the long call decreases with a higher exercise price while the payoff of the long put increases with ah higher exercise price. The value of the long call option will therefore increase when the interest rate increases while the payoff of the long put option will decrease.

$$
\begin{aligned}
f_t^{long\ call} &= S_t - PV(K) \\
f_t^{long\ put} &= PV(K) - S_t
\end{aligned}
$$

- Next, we investigate the impact of the volatility σ on the value of options.
  - Consider a scenario where the price of the asset, underlying an option contract, is equal to \$100. During the next period the price of the underlying could decline to \$90 or could increase to \$110, with equal probabilities. Consider also a long European call option on this underlying with an exercise price of \$100 that expires at the end of the next period. If the price of the underlying rises to \$110, the payoff of the long call option will be equal to \$10. If the price of the underlying rises to \$90, the payoff of the long call option will be equal to \$0. The expected return in this case is equal to \$5.
  - Consider a second scenario where the price of the underlying could decline to \$80 or could increase to \$120 during the next period, with equal probabilities. If the price of the underlying rises to \$120, the payoff of the long call option will be equal to \$20. If the price of the underlying rises to \$90, the payoff of the long call option will be equal to \$0. The expected return in this case is equal to \$10.
  - From this example, it is clear that a higher volatility in the price of the underlying translates to a higher expected payout for the long call option. Options are volatility-driven instruments. The long position in an option contract will gain if the volatility in the price of the underlying asset increases. The short position will gain if the volatility of the underlying asset decreases.

- In the previous chapter we learned that the value of future and forward contracts reflected views on the direction of the market price of the underlying. The value of the option contract however, reflects both the view of the market on the direction and the volatility of the price of the underlying asset.
- The table below summarizes the outlook on the volatility and the direction of the market for the underlying for different types of options.

	Bullish on direction	Bearish on direction
Bullish on volatility	Long call option	Long put option
Bearish on volatility	Short put option	Short call option

- Finally, we investigate the influence of dividends on the value of options.
  - The price of the asset, underlying the option contract declines after a dividend has been paid out.
  - A decline in the value of the underlying translates to a decline in the value of the long call option and a rise in the value of the long put option.
  - The option style is of no importance in this case (i.e. whether the option is European-style or American-style.)
- We can now summarize the influence of the different variables on the value of options. This is done in the table below for the long call and the long put option for European-style options and American-style options.

	$f_t^{Eur.\ long\ call}$	$f_t^{Am.\ long\ call}$	$f_t^{Eur.\ long\ put}$	$f_t^{Am.\ long\ put}$
S_t	+	+	−	−
K	−	−	+	+
τ	+	+	?	+
r	+	+	−	−
σ	+	+	+	+
D	−	−	+	+

12 The binomial option pricing model

12.1 Discrete and continuous time

- A discrete-time model divides time into small time intervals Δt. It models a process between these time intervals, i.e. at different fixed points in time. We could also model time in a continuous manner. This is done by dividing time into infinitesimally small increments dt. The process is modeled between these infinitesimally small time increments.
- For practical purposes, it will sometimes be necessary to discretize time when using using a continuous time model. In that case the idea of discrete time is superimposed on the continuous time model. The continuous time process is then measured between different intervals in time i.e. at fixed points in time.

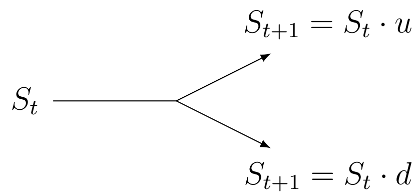
12.2 Models for the price behavior of the underlying

- The goal of this chapter is to construct a model for option pricing. Options are derivatives; which means that their value is derived from the value of an underlying. We will therefore have to model the value of the underlying first, before a model of for the value of the options themselves can be constructed.
- First, we model time and the price of the underlying in a discrete manner. The price of the underlying is considered to be a stochastic variable. All of this means that the price of the underlying will evolve to one of a discrete number of prices by the start of the next time period with a certain probability. The price thus follows a discrete probability distribution. A model where there are only two possible prices by the start of the next period is called a binomial model.
- A binomial model for the price of the underlying could define that the price of the underlying either increases to a certain level with a certain probability or decreases to a certain level with the inverse probability of the first event.

  The possible prices at the start of the next period are obtained by multiplying the current price with either a positive factor u or a negative factor d, where u stands for upstate and d stands for downstate. The binomial model therefore is also a multiplicative model.

In a mathematical and graphical representation, this becomes:

$$S_{t+1} = \begin{cases} P(u) : S_t \cdot u \\ P(d) : S_t \cdot d \end{cases}$$



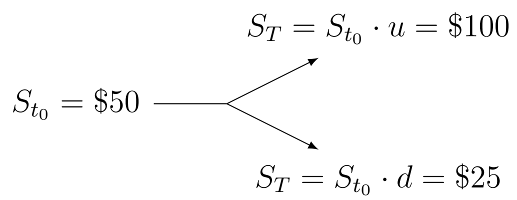
- Suppose the price of the underlying could also remain constant. In that case there are three different possible states of the world defined by our model. Such a model is called a trinomial model. The basic idea however, remains the same. Id est the model defines only a limited or discrete number of possibles states of the world or outcomes.
- Another approach would be to model the price of the underlying in a continuous manner. This means that the price of the underlying evolves according to a continuous distribution. In that case there are an infinite number of possible values which the price of the underlying could take on. The model defines a probability for any range of possible values for the price of the underlying.

  The evolution of the value of the underlying asset could for instance be modelled by compounding the value of the asset with a return that is randomly drawn from a standard normal distribution Mathematically:

$$S_{t+1} = S_t \cdot e^r$$
$$r \sim \mathcal{N}(0, 1)$$

12.3 A replicating portfolio for the European-style long call option

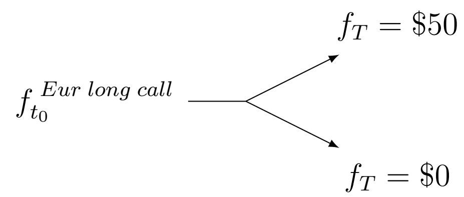
- This section looks into how a replicating portfolio can be used in pricing a European-style call option. Id est we model the price of the European-style long call option, in a discrete number of states of the world, by using a replicating portfolio.
- We illustrate the basic concepts using the following example.
  - Consider a binomial model for the price/value of a certain asset. The price of the asset today i.e. at time t_0 is equal to \$50. By the end of the current period i.e. at time T, the price of the asset will have either increased to \$100 with a certain probability p or will have decreased to \$25 with the inverse probability $(1 - p)$. We can represent this binomial model in a graphical manner.



– Next, we derive the value of a European long call option in these states of the world. Suppose this call option has an exercise price of \$50 and matures at the end of the current period i.e. at time T. The payoff of this long call option will therefore be either \$0 or \$50.

  We again illustrate this model in a mathematical and graphical way.

$$f_T^{Eur.\ long\ call} = \begin{cases} \$50, & S_T = \$100 \\ \$0, & S_T = \$25 \end{cases}$$



- We will not present a formal proof that shows how we can construct a replicating portfolio for the European long call option. We will only illustrate such a replicating portfolio in the example below.

  – Consider the portfolio we constructed below. This portfolio is constructed in a very particular way so that it does not provide any cashflow at time T. The value of the portfolio at time t_0 is therefore equal to zero. The portfolio consists of three short call options, two units of the underlying and a loan.

	t_0	$T : S_T = \$25$	$T : S_T = \$100$
Three short call options	$+3 \cdot f_{t_0}^{Eur.\ long\ call}$	\$0	$\$-150$
Two units long spot	$-\$100$	$+\$50$	$+\$200$
Loan	$+\$40$	$-\$50$	$-\$50$
Total	$+3 \cdot f_{t_0}^{Eur.\ long\ call} - \$60$	0	0

  – Notice that the portfolio consisting of only the short positions in the European call option and the long spot positions, is equivalent with a riskless investment. The payoff of this portfolio at maturity is fixed under all circumstances. The original portfolio is therefore a combination of a riskless investment with a riskless loan. The two positions cancel each other out.

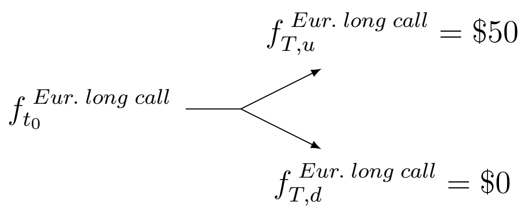
	t_0	$T : S_T = \$25$	$T : S_T = \$100$
Three short call options	$+3 \cdot f_{t_0}^{Eur.\ long\ call}$	$0	$ − 150
Two units long spot	−$100	+$50	+$200
Total	$ − 40	$ + 50	$ + 50

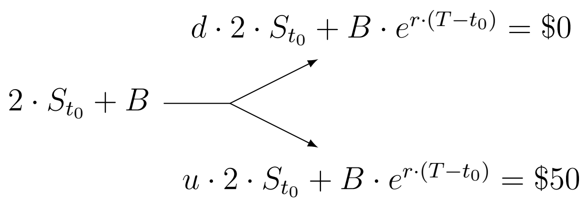
– In the first portfolio, the risk from the short call options is hedged by going long spot in the underlying. The cost of this operation is covered by premia from the short call options which are matched by entering a loan. This strategy is called a riskless hedge.

– Notice that the portfolio consisting of only the long spot positions and the loan yields the opposite cashflows of the position in the short call options. From this, it is clear that three long call options can be replicated by buying two units of the underlying and by entering the loan. The long call options are thus equivalent with a portfolio that contains the underlying which is leveraged by entering a loan.

	t_0	$T : S_T = \$25$	$T : S_T = \$100$
Two units long spot	−$100	+$50	+$200
Loan	+$40	−$50	−$50
Total	$ − 60	$0	$ + 150

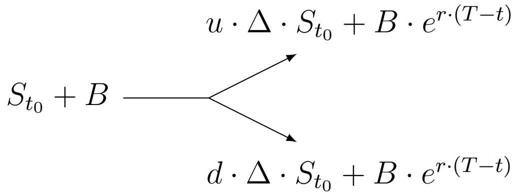
– This last portfolio is therefore a replicating portfolio for three long European-style call options. The cashflows yielded by this replicating portfolio are equal to the cashflows of the long call options, in all states of the world. The figures below illustrate the possible states of the world for the European-style long call option and its replicating portfolio.





- We successfully replicated the long call option, using a riskless loan and a long position in a number of units of the underlying.
- In general, we can create a replicating portfolio for a call option by taking a long or short position in the underlying asset and by simultaneously entering a riskless investment or a riskless loan.
- Suppose Δ represents the number of units of the underlying asset in our portfolio and B represents the the value of the loan. Using this notation, we can generalize our findings. Mathematically and graphically, this becomes:

$$\begin{cases} f_{t,d}^{Eur.\ long\ call} & = d \cdot \Delta \cdot S_{t_0} + B \cdot e^{r \cdot (T-t)} \\ f_{t,u}^{Eur.\ long\ call} & = u \cdot \Delta \cdot S_{t_0} + B \cdot e^{r \cdot (T-t)} \end{cases}$$



- It is clear that the value of the replicating portfolio has to be equal to the value of the long call option, in all possible states of the world. We can therefore derive the following system of equations:

$$\begin{cases} f_{T,d}^{Eur.\ long\ call} & = d \cdot \Delta \cdot S_{t_0} + B \cdot e^{r \cdot (T-t_0)} \\ f_{T,u}^{Eur.\ long\ call} & = u \cdot \Delta \cdot S_{t_0} + B \cdot e^{r \cdot (T-t_0)} \end{cases}$$

- From this, we derive an expression for Δ i.e. the number of units of the underlying asset in our portfolio. Mathematically:

$$\begin{aligned} f_{T,u}^{Eur.\ long\ call} - f_{T,d}^{Eur.\ long\ call} &= u \cdot \Delta \cdot S_{t_0} + B \cdot e^{r \cdot (T-t_0)} - d \cdot \Delta \cdot S_{t_0} + B \cdot e^{r \cdot (T-t_0)} \\ &= \Delta(uS_{t_0} - dS_{t_0}) \end{aligned}$$

$$\Delta = \frac{f_{T,u}^{Eur.\ long\ call} - f_{T,d}^{Eur.\ long\ call}}{uS_{t_0} - dS_{t_0}}$$

- The original replicating portfolio contained three European call options which were hedged with two units of the underlying. The delta in this case is equal to $\frac{2}{3}$. Therefore, Δ seems to tell us something about the number of units of the underlying we need, to hedge the option position.

- We substitute the value for Δ in one of the equations for the payoff of the portfolio, at time T, to derive an expression for the principal value of the loan B. Mathematically, this becomes:

$$\begin{aligned}
f_{T,d}^{\text{Eur. long call}} &= d \cdot \Delta \cdot S_{t_0} + B \cdot e^{r\cdot(T-t_0)} \\
&= d \cdot \frac{f_{T,u}^{\text{Eur. long call}} - f_{T,d}^{\text{Eur. long call}}}{uS_{t_0} - dS_{t_0}} \cdot S_{t_0} + B \cdot e^{r\cdot(T-t_0)} \\
&= \frac{d \cdot f_{T,u}^{\text{Eur. long call}} - d \cdot f_{T,d}^{\text{Eur. long call}}}{u-d} + B \cdot e^{r\cdot(T-t_0)}
\end{aligned}$$

$$\begin{aligned}
B \cdot e^{r\cdot(T-t_0)} &= f_{T,d}^{\text{Eur. long call}} - \frac{d \cdot f_{T,u}^{\text{Eur. long call}} - d \cdot f_{T,d}^{\text{Eur. long call}}}{u-d} \\
&= \frac{u-d}{u-d} \cdot f_{T,d}^{\text{Eur. long call}} - \frac{d \cdot f_{T,u}^{\text{Eur. long call}} - d \cdot f_{T,d}^{\text{Eur. long call}}}{u-d} \\
&= \frac{u \cdot f_{T,d}^{\text{Eur. long call}} - d \cdot f_{T,d}^{\text{Eur. long call}} - d \cdot f_{T,u}^{\text{Eur. long call}} + d \cdot f_{T,d}^{\text{Eur. long call}}}{u-d} \\
&= \frac{u \cdot f_{T,d}^{\text{Eur. long call}} - d \cdot f_{T,u}^{\text{Eur. long call}}}{u-d}
\end{aligned}$$

$$B = \frac{u \cdot f_{T,d}^{\text{Eur. long call}} - d \cdot f_{T,u}^{\text{Eur. long call}}}{(u-d) \cdot e^{r(T-t_0)}}$$

- We have derived an expression for the number of units of the underlying Δ and for the principal value of the loan B. We can now derive the value of the call option at time t_0:

$$\begin{aligned}
f_{t_0}^{\text{Eur. long call}} &= S_{t_0} \cdot \Delta + B \\
\Delta &= \frac{f_{T,u}^{\text{Eur. long call}} - f_{T,d}^{\text{Eur. long call}}}{uS_{t_0} - dS_{t_0}} \\
B &= \frac{u \cdot f_{T,d}^{\text{Eur. long call}} - d \cdot f_{T,u}^{\text{Eur. long call}}}{(u-d) \cdot e^{r(T-t_0)}}
\end{aligned}$$

- We now only have to substitute the expressions for Δ and B in the formula for the long European call option. Mathematically, this becomes:

$$
\begin{aligned}
f_{t_0}^{\textit{Eur. long call}} &= \Delta \cdot S_{t_0} + B \\
&= \frac{f_{T,u}^{\textit{Eur. long call}} - f_{T,d}^{\textit{Eur. long call}}}{uS - dS} \cdot S_{t_0} + \frac{u \cdot f_{T,d}^{\textit{Eur. long call}} - d \cdot f_{T,u}^{\textit{Eur. long call}}}{(u-d) \cdot e^{r(T-t_0)}} \\
&= \frac{R \cdot (f_{T,u}^{\textit{Eur. long call}} - f_{T,d}^{\textit{Eur. long call}})}{R \cdot (u-d)} + \frac{u \cdot f_{T,d}^{\textit{Eur. long call}} - d \cdot f_{T,u}^{\textit{Eur. long call}}}{(u-d) \cdot R} \\
&= \frac{R \cdot f_{T,u}^{\textit{Eur. long call}} - R \cdot f_{T,d}^{\textit{Eur. long call}} + u \cdot f_{T,d}^{\textit{Eur. long call}} - d \cdot f_{T,u}^{\textit{Eur. long call}}}{R \cdot (u-d)} \\
&= \frac{\frac{R-d}{u-d} \cdot f_{T,u}^{\textit{Eur. long call}} + \frac{u-R}{u-d} \cdot f_{T,d}^{\textit{Eur. long call}}}{R}
\end{aligned}
$$

- We can clearly see that the value of the European call option is equal to the sum of two discounted cashflows.
- Note that both fractions in the numerator are positive and sum up to one. These fractions can therefore be interpreted as probabilities. We can rewrite the expression for the option price, interpreting these fractions as probabilities. Mathematically, this becomes:

$$
\begin{aligned}
p &= \frac{R-d}{u-d} \\
(1-p) &= \frac{u-R}{u-d}
\end{aligned}
$$

$$
f_{t_0}^{\textit{Eur. long call}} = \frac{p \cdot f_{t,u} + (1-p) \cdot f_{t,d}}{R}
$$

- The formula above tells us to take the average payout of a risky instrument and to discount the payout using the riskless interest rate. Something seems wrong here. Why would we want to discount the cashflow of a risky instrument, using a riskless interest rate? To see why this makes sense, consider three different investors:
  - A risk-averse investor who expects to receive the risk free interest RF rate plus a risk premium RP on his investment. This investor is not willing to take a risk without compensation.
  - A risk-neutral investor who expects to receive the risk-free interest rate RF on his investment. This investor does not care about the riskiness of his investment.
  - A risk-seeking investor who to expects to receive the risk-free interest RF minus a risk premium RP on his investment. This investor is willing to pay a risk premium.

- A risk-neutral investor would look for any investment that satisfies the following condition:

$$
\begin{aligned}
p \cdot u \cdot S_{t_0} + (1-p) \cdot d \cdot S_{t_0} &= S_{t_0} \cdot R \\
p \cdot u + d - p \cdot d &= R \\
p \cdot (u-d) &= R - d \\
p &= \frac{R-d}{u-d}
\end{aligned}
$$

- The probability p is defined as a risk-neutral probability i.e. the probability that a risk-neutral investor would ascribe to the upstate of the underlying.
- The formula uses risk-neutral probabilities. We are evaluating the option from a risk-neutral perspective. We can therefore discount the payout at the riskless rate. Whether we use the point of view of the risk-neutral investor, the risk-averse investor or the risk-seeking investor does not matter in determining the price of the option. However, when using the probabilities ascribed to the possible states of the world from another investor's perspective, the interest rate, used for discounting in the option pricing formula, will not be equal to the risk-free rate. Id est the interest rate that has to be used is unknown. The price is unambiguously determined. If the option price deviates from the theoretical price, given by the expression we derived, arbitrage is possible. Using the risk-neutral probabilities is easier however, because we can then use the risk-free interest rate to discount the possible cashflows.

12.4 The two-period binomial option pricing model

- In this section, we take further steps towards generalizing our option pricing model by increasing the number of periods from one to two. Where a one-period model yields two possible outcomes with two corresponding probabilities, a two-period model yields three possible outcomes with three corresponding probabilities. The two-period model therefore offers a better view of the possible outcomes and the attached probabilities. We get a more refined view on what can happen to the underlying and hence what can happen to the option price at maturity.
- Building a tree can be done using one of two different methods. The first method is called branching out, the result is a non-recombining tree. After n steps, the tree contains 2^n nodes. The second method is called recombining, the result is a recombining tree. This tree is the result of a multiplicative model. After n steps, the tree contains n nodes and 2^n different paths. We will build a recombining tree.
- Assume that the volatility is constant. In that case, the probability of the upstate is always equal to p and the probability of the downstate is always equal to $(1-p)$, across the different periods.
- Consider the expression we derived for the option price in the previous section:

$$f_{t_0}^{Eur.\ long\ call} = \frac{p \cdot f_{T,u} + (1-p) \cdot f_{T,d}}{R}$$

- This formula can be adapted for the case where there are two periods instead of one. The adapted formula calculates the value of an option at the start of a period based on the option prices at the end of that period. Mathematically, this becomes:

$$f_{t-1}^{Eur.\ long\ call} = \frac{p \cdot f_{t,u} + (1-p) \cdot f_{t,d}}{R}$$

- If we know the distribution of the option prices at time t, we can derive the distribution of the option prices at time $t-1$. Id est if we know the distribution of the option prices in one period, we can derive the distribution of the option prices in the previous period.
- Determining the price of an option in a given period is thus a recursive process which uses the formula above. In the first step we grow a tree of the value of the underlying, given the factors u and d. We can then determine the value of the option at maturity T which is equal to its payoff. The calculation of the payoff of the option is based on the strike price and the value of the underlying asset at maturity, both of which are known. Furthermore, we can apply the formula above to determine the price of the option at time $T-1$. When the price of the option at $T-1$ is know, the formula can again be used to determine the value at time $T-2$ et cetera. Mathematically, this becomes:

$$\begin{cases} f_{(t-1),u}^{Eur.\ long\ call} &= \frac{p \cdot f_{t,uu} + (1-p) \cdot f_{t,ud}}{R} \\ f_{(t-1),d}^{Eur.\ long\ call} &= \frac{p \cdot f_{t,ud} + (1-p) \cdot f_{t,dd}}{R} \end{cases}$$

- In this section we specifically considered a two-period model. We will therefore derive the value of the European-style long call option at time t_0 in the case of a two period model. Mathematically, this becomes:

$$
\begin{aligned}
f_{T-2}^{Eur.\ long\ call} &= \frac{p \cdot f_{T-1,u}^{Eur.\ long\ call} + (1-p) \cdot f_{T-1,d}^{Eur.\ long\ call}}{R} \\
&= \frac{p \cdot \left[\frac{p \cdot f_{T,uu}^{Eur.\ long\ call} + (1-p) \cdot f_{T,ud}^{Eur.\ long\ call}}{R}\right] + (1-p) \cdot \left[\frac{p \cdot f_{T,ud}^{Eur.\ long\ call} + (1-p) \cdot f_{T,dd}^{Eur.\ long\ call}}{R}\right]}{R} \\
&= \frac{p^2 \cdot f_{T,uu}^{Eur.\ long\ call} + 2 \cdot (1-p) \cdot p \cdot f_{T,ud}^{Eur.\ long\ call} + (1-p)^2 \cdot f_{T,dd}^{Eur.\ long\ call}}{R^2}
\end{aligned}
$$

From this we can see that the European-style long call option is the discounted value, over two periods, of the potential cashflows which are multiplied with their respective probabilities.

12.5 The three-period binomial option pricing model

- In the previous section, we increased the number of periods in the binomial model from one to two. We derived that the value of the European long call option is the discounted value, over two periods, of the potential cashflows multiplied with their respective probabilities.
- Consider a binomial model with three different periods instead of two. We can easily see that the value of the European call is given by:

$$\begin{aligned} f_{T-3}^{\text{Eur. long call}} = \Big[& p^3 \cdot f_{T,uuu}^{\text{Eur. long call}} \\ & + 3 \cdot p^2(1-p) \cdot f_{T,uud}^{\text{Eur. long call}} \\ & + 3 \cdot p \cdot (1-p)^2 \cdot f_{T,udd}^{\text{Eur. long call}} \\ & + (1-p)^3 \cdot f_{T,ddd}^{\text{Eur. long call}} \Big] / R^3 \end{aligned}$$

- In this case there are a total of $2^3 = 8$ paths. We are interested in calculating the total number of paths that lead to the same outcome. I.e. we want to calculate how many paths lead to the same leaf node. Every combination of k successes or upticks in a collection of n trials or paths, leads to the same outcome. The number of combinations of k upticks in a collection of n paths, is given by:

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

- The formula for the value of the European call option, three periods prior to the maturity of the option T, can therefore be written in the following fashion:

$$\begin{aligned} f_{T-3}^{\text{Eur. long call}} = \Big[& p^3 \cdot f_{T,uuu}^{\text{Eur. long call}} \cdot \binom{3}{3} \\ & + p^2(1-p) \cdot f_{T,uud}^{\text{Eur. long call}} \cdot \binom{3}{2} \\ & + p \cdot (1-p)^2 \cdot f_{T,udd}^{\text{Eur. long call}} \cdot \binom{3}{1} \\ & + (1-p)^3 \cdot f_{T,ddd}^{\text{Eur. long call}} \cdot \binom{3}{0} \Big] / R^3 \end{aligned}$$

12.6 The multiperiod binomial option pricing model

- We can generalize the binomial option pricing model for an arbitrary number of periods n and a number of successes j. Mathematically, this becomes:

$$f_{T-n}^{Eur.\ long\ call} = \frac{\sum_{j=0}^{n} \binom{n}{j} \cdot p^j (1-p)^{n-j} \cdot \left[max\left(0, S_{t_0} \cdot u^j \cdot d^{(n-j)} - K\right)\right]}{R^n}$$

- In summary, we derived a formula that allows us to recursively determine the value of an option, n periods prior to the last period. Graphically, this corresponds to folding the tree with the option prices from the back towards the front. Mathematically, this corresponds to substituting the formula for the option price of time t in the formula for the option price of time $t-1$.
- Notice that this method of option pricing is only possible if there are no early exercise features. This is because this method assumes that the final cashflows and their corresponding probabilities are given. Id est we take the final cashflows and their corresponding probabilities as given. From this we compute the expected cashflow, which is discounted back to today.
- Notice the maximum operator in the expression above. We would simplify the expression for the binomial option pricing model by reformulating the expression without this operator. The maximum operator reflects the fact that the option will only be exercised when the value of the payoff is positive. For the option to be in the money, there need to be a yet unknown number of upticks a. We will now derive this minimum number of upticks a. Mathematically, this becomes:

$$\begin{aligned} u^a \cdot d^{n-a} \cdot S - K &\geq 0 \\ u^a \cdot d^{(n-a)} &\geq \frac{K}{S} \\ a \cdot ln(u) + ((n-a)) \cdot ln(d) &\geq ln(\frac{K}{S}) \\ a \cdot (ln(\frac{u}{d})) + ln(d^n) &\geq ln(\frac{K}{S}) \\ a &\geq \frac{ln(\frac{K}{S}) - ln(d^n)}{ln(\frac{u}{d})} \end{aligned}$$

- Notice that the factor in square brackets in the numerator of the expression is equal to zero when j is smaller than a. This implies that all terms for which $j < a$ are equal to zero and will not contribute to the sum in the numerator. We can therefore start the sum for $j = a$. Doing so will make the maximum operator redundant because the payoff will always be greater or equal to zero for $j \geq a$. Mathematically, this adapted version of the binomial option pricing model becomes:

$$f_{T-n}^{Eur.\ long\ call} = \frac{\sum_{j=a}^{n} \binom{n}{j} \cdot p^j \cdot (1-p)^{(n-j)} \cdot \left[u^j \cdot d^{(n-j)} \cdot S_{t_0} - K\right]}{R^n}$$

12.7 Interpreting the binomial option pricing model

- Notice that we can split the equation for the binomial option pricing model into two different parts. Mathematically, this becomes:

$$f_{T-n}^{Eur.\ long\ call} = \frac{\sum_{j=a}^{n} \binom{n}{j} \cdot p^j \cdot (1-p)^{(n-j)} \cdot \left[u^j \cdot d^{(n-j)} \cdot S_{t_0} - K\right]}{R^n}$$

$$= \frac{\sum_{j=a}^{n} \left[\binom{n}{j} \cdot p^j \cdot (1-p)^{(n-j)} \cdot u^j \cdot d^{(n-j)} \cdot S_{t_0}\right]}{R^n} - \frac{K}{R^n} \cdot \left[\sum_{j=a}^{n} \binom{n}{j} \cdot p^j \cdot (1-p)^{n-j}\right]$$

- Consider the first term in the expression which reflects the present value of the expected cashflows from the asset underlying the option.

$$\frac{\sum_{j=0}^{n} \binom{n}{j} \cdot p^j (1-p)^{n-j} \cdot u^j \cdot d^{(n-j)} \cdot S_{t_0}}{R^n}$$

  - The following factor reflects the probability of being in the money.

$$\sum_{j=a}^{n} \binom{n}{j} \cdot p^j (1-p)^{(n-j)}$$

  - While the following factor reflects the potential prices of the underlying when the option is in the money.

$$\sum_{j=a}^{n} u^j \cdot d^{n-j} \cdot S_{t_0}$$

- Consider the second term in the expression which reflects the present value of the expected cost that is incurred by exercising the option, when the option is in the money.

$$-\frac{K}{R^n} \cdot \left[\sum_{j=a}^{n} \binom{n}{j} \cdot p^j \cdot (1-p)^{n-j}\right]$$

  - The first factor reflects the present value of the strike price.

$$-\frac{K}{R^n}$$

– The second factor reflects the sum of all probabilities of paths where j is greater than or equal to a id est the cumulative probability that the option is in the money.

$$\sum_{j=a}^{n} \binom{n}{j} \cdot p^j \cdot (1-p)^{n-j}$$

- We can clearly see that the formula for the binomial option pricing model compares the cost of exercising the option with the benefit of exercising the option at a given point in time. The binomial option pricing model therefore gives the net benefit of exercising the option.

12.8 Binomial option pricing: dividends

- We now investigate how the binomial option pricing model can be adapted to the case where the asset underlying the option contract, provides a dividend.
  – First, recall the expression for the binomial option pricing model. Mathematically:

$$f_{t_0}^{\,Eur.\ long\ call} = \frac{p \cdot f_{T,u}^{\,Eur.\ long\ call} - (1-p) \cdot f_{T,d}^{\,Eur.\ long\ call}}{R}$$

Where p is defined in the following manner:

$$p = \frac{R-d}{u-d}$$

  – Suppose the asset, underlying the option contract provides a dividend during the lifetime of the option contract. We could include this dividend in the model by adjusting the drift rate. In the risk-neutral world, the risk-free interest rate is the drift rate of the stock price. Mathematically, this becomes:

$$p = \frac{e^{(r-\gamma)\cdot\tau} - d}{u-d}$$
$$p = \frac{a-d}{u-d}$$

Where R and a are defined in the following manner:

$$\begin{aligned} R &= 1 + r \\ &= e^{r} \end{aligned}$$

$$\begin{aligned} a &= (r-\gamma)\cdot\tau \\ &= e^{(r-\gamma)\cdot\tau} \end{aligned}$$

12.9 Delta hedging

- Recall from the first example in this chapter that a position in the underlying could be hedged by entering a position in short call options. The delta Δ expressed the number of short call options which were required to offset the position in the underlying.
  - Recall the expression we derived for the delta Δ. Mathematically:

$$\Delta = \frac{f_{T,u} - f_{T,d}}{uS_{t_0} - dS_{t_0}}$$

  - In general, the delta Δ expresses the relationship between price changes in the underlying and price changes in the option contract. Mathematically:

$$\Delta = \frac{\delta f}{\delta S}$$

  - The delta for a long call option is always positive and has a value between zero and one. If the value of the underlying increases, the value of the long call option also increases.
  - If the long call option is very deep in the money, the delta will be equal to one. Id est if the value of the underlying goes up by a certain amount, the value of the long call option will go up by the same amount. If the long call option is very deep out of the money, the delta will be equal to zero. Id est fluctuations in the price of the underlying have no real impact on the value of the long call option.
  - Notice that the delta can change over time. If we are hedging over a certain period of time, we will have to adjust the hedge over time. When the delta changes, the relationship between price changes in the option and price changes in the underlying changes.
  - For forward contracts, the delta remains one over the whole time period. It is therefore possible to set up a hedge with future contracts over a certain time period without having to adjust the hedge during this period.
  - In order to hedge with options, we will need a dynamic strategy. When hedging with call options for example, we will have to adjust the number of units in the underlying vis â vis the written call options on this underlying.

12.10 The multiperiod binomial option pricing model in a spreadsheet program

- It is pretty easy to implement a binomial option pricing model in a spreadsheet program. We consider a two-period binomial option pricing model for a long call option. The figure below shows how to construct a binomial tree for such a case. On the left side we outline the current price of the underlying S_{t_0}, the strike price of the option K, the factors u and d and the gross interest rate R. The probability p can be easily computed as $\frac{R-d}{u-d}$. The probability $(1-p)$ is of course the inverse probability.
- The first step entails constructing the binomial tree for the price of the underlying. The upstate price of the underlying after one period is acquired by multiplying the price of the underlying in the current period with u. The downstate price of the underlying after one period is acquired by multiplying the price of the underlying in the current period with d.

	A	B
1		
2		
3		
4	**St0**	50
5	**K**	50
6	**u**	1.765
7	**d**	0.566572238
8	**R**	111.80%
9		
10	**p**	=(B8-B7)/(B6-B7)
11	**(1-p)**	0.539874009
12		

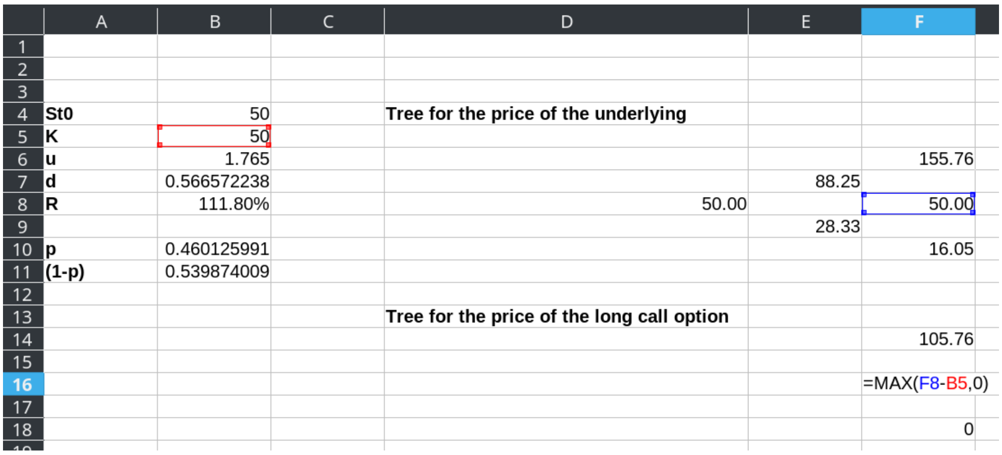
Figure 12.1: Calculating the probabilities p and $(1-p)$.

	A	B	C	D	E	F
1						
2						
3						
4	**St0**	50		**Tree for the price of the underlying**		
5	**K**	50				
6	**u**	1.765				155.76
7	**d**	0.566572238			=D8*B6	
8	**R**	111.80%		50.00		50.00
9					28.33	
10	**p**	0.460125991				16.05
11	**(1-p)**	0.539874009				
12						

Figure 12.2: Calculating the upstate price of the underlying based on the price of the previous period and the factor u.

- The second step entails calculating the payoff of the option at maturity T, in all possible states of the world. The figure below shows how this is done in a spreadsheet program. The payoff function of a long call option is given by:

$$f_T = max\ (S_T - K, 0)$$

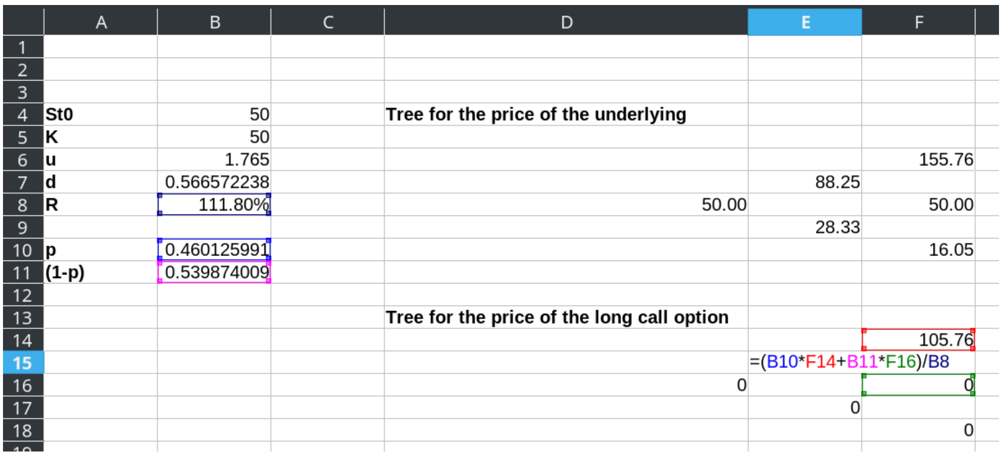


	A	B	C	D	E	F
1						
2						
3						
4	St0	50		Tree for the price of the underlying		
5	K	50				
6	u	1.765				155.76
7	d	0.566572238			88.25	
8	R	111.80%		50.00		50.00
9					28.33	
10	p	0.460125991				16.05
11	(1-p)	0.539874009				
12						
13				Tree for the price of the long call option		
14						105.76
15						
16						=MAX(F8-B5,0)
17						
18						0

Figure 12.3: Calculating the payoff of the long call option at maturity T.

- We now have the price of the long call option at maturity T. It is therefore possible to calculate the price of the call option, one period prior using the formula of the binomial option pricing model. The figure below shows how this is done in a spreadsheet program. Mathematically, this becomes:

$$f_{t-1}^{Eur.\ long\ call} = \frac{p \cdot f_{t,u} + (1-p) \cdot f_{t,d}}{R}$$



	A	B	C	D	E	F
1						
2						
3						
4	St0	50		Tree for the price of the underlying		
5	K	50				
6	u	1.765				155.76
7	d	0.566572238			88.25	
8	R	111.80%		50.00		50.00
9					28.33	
10	p	0.460125991				16.05
11	(1-p)	0.539874009				
12						
13				Tree for the price of the long call option		
14						105.76
15					=(B10*F14+B11*F16)/B8	
16				0		0
17					0	
18						0

Figure 12.4: Calculating the value of the option one period prior to the maturity T.

- Notice that this representation of the binomial tree does not translate well to a spreadsheet. We will therefore propose another representation for the tree which is shown in the figure below. Each row represents the price of the underlying at different points in time. The first rows represents a geometric sequence of prices of the underlying for different points in time where the common ratio is equal to u. The second row is equal to the first row multiplied with the factor $\frac{d}{u}$, the third row is equal to the second row multiplied with the factor $\frac{d}{u}$, et cetera. Notice that each new row moves up one column to the right. The binomial tree for the option prices can be derived in the same manner as before.

	A	B	C	D	E	F
1						
2						
3						
4	**St0**	50		**Tree for the price of the underlying**		
5	**K**	50		50	88.25	155.76125
6	**u**	1.765			28.33	50
7	**d**	0.566572238				=F6*($B$7/$B$6)
8	**R**	111.80%				
9						
10	**p**	0.460125991				
11	**(1-p)**	0.539874009				

Figure 12.5: An alternative, more spreadsheet-friendly representation of the binomial tree for the price of the underlying.

	A	B	C	D	E	F
1						
2						
3						
4	**St0**	50		**Tree for the price of the underlying**		
5	**K**	50		50	88.25	155.76125
6	**u**	1.765			28.33	50
7	**d**	0.566572238				16.05
8	**R**	111.80%				
9						
10	**p**	0.460125991				
11	**(1-p)**	0.539874009				
12						
13				**Tree for the price of the long call option**		
14						
15						
16						=MAX(F5-$B$5)
17						0
18						-33.94979496

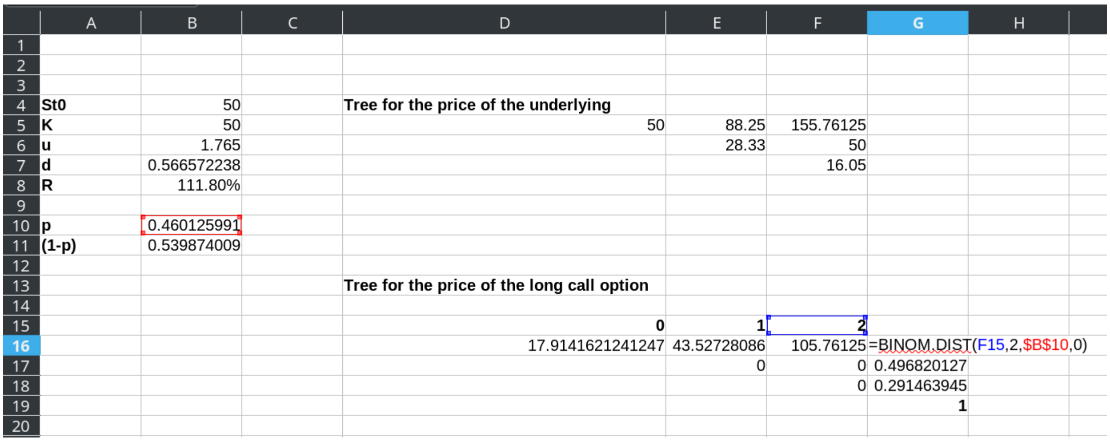
Figure 12.6: Calculating the price of the option at maturity using the alternative representation.

	A	B	C	D	E	F
1						
2						
3						
4	**St0**	50		**Tree for the price of the underlying**		
5	**K**	50		50	88.25	155.76125
6	**u**	1.765			28.33	50
7	**d**	0.566572238				16.05
8	**R**	111.80%				
9						
10	**p**	0.460125991				
11	**(1-p)**	0.539874009				
12						
13				**Tree for the price of the long call option**		
14						
15						
16				=($B$10*E16+$B$11*E17)/$B$8	43.52728086	105.76125
17					0	0
18						0

Figure 12.7: Calculating the price of the option before maturity using the alternative representation.

- We can also calculate the probability that we will end up in a specific leaf node i.e. the probability that the option price is equal to a certain value. This done using the formula for the binomial distribution. In our spreadsheet we can use the "BINOM.DIST" function. Mathematically:

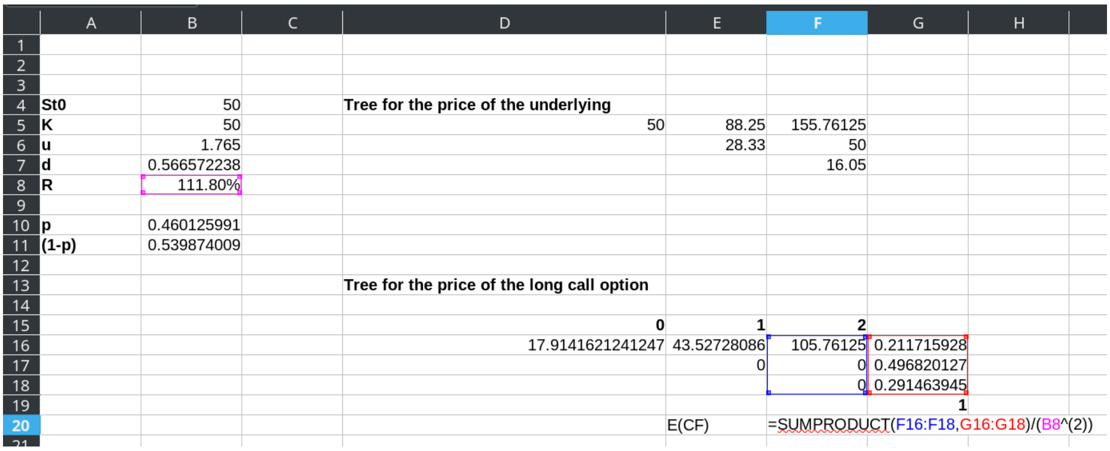
$$P_x = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}$$



	A	B	C	D	E	F	G	H
1								
2								
3								
4	St0	50		Tree for the price of the underlying				
5	K	50		50	88.25	155.76125		
6	u	1.765			28.33	50		
7	d	0.566572238				16.05		
8	R	111.80%						
9								
10	p	0.460125991						
11	(1-p)	0.539874009						
12								
13				Tree for the price of the long call option				
14								
15				0	1	2		
16				17.9141621241247	43.52728086	105.76125	=BINOM.DIST(F15,2,$B$10,0)	
17					0	0	0.496820127	
18						0	0.291463945	
19							1	
20								

Figure 12.8: Calculating the probability for each leaf node in the binomial tree.

- Finally, we can compute the expected cashflow for the long call option. This is done by taking the sum of the product of every cashflow that corresponds to a leaf node and its corresponding probability. We then discount this expected cashflow back to the present time to obtain the price of the option today. The figure below illustrates how this can be done in a spreadsheet program.



	A	B	C	D	E	F	G	H
1								
2								
3								
4	St0	50		Tree for the price of the underlying				
5	K	50		50	88.25	155.76125		
6	u	1.765			28.33	50		
7	d	0.566572238				16.05		
8	R	111.80%						
9								
10	p	0.460125991						
11	(1-p)	0.539874009						
12								
13				Tree for the price of the long call option				
14								
15				0	1	2		
16				17.9141621241247	43.52728086	105.76125	0.211715928	
17					0	0	0.496820127	
18						0	0.291463945	
19							1	
20					E(CF)	=SUMPRODUCT(F16:F18,G16:G18)/(B8^(2))		

Figure 12.9: Calculating the present value of the expected cashflow for the long call option.

13 Continuous time mathematics primer

- Recall the formula for the value of the European call option:

$$f_{t_0}^{Eur.\ long\ call} = \frac{\sum_{j=a}^{n} \binom{n}{j} \cdot p^j \cdot (1-p)^{n-j} \cdot u^j \cdot d^{n-j}}{R^n} \cdot S_T$$
$$= \Big[\sum_{j=a}^{n} \binom{n}{j} \cdot p^j \cdot (1-p)^{n-j} \Big] \cdot \frac{K}{R^n}$$

- Our goal is to compute the limiting case for the binomial option pricing model, where the number of periods goes to infinity. Id est, we let the number of steps in the tree approach infinity. Mathematically, this becomes:

$$\lim_{n \to \infty} (f_{t_0}^{Eur.\ long\ call})$$

- First and foremost, we repeat some basic mathematical concepts. We will use these concepts later on in this chapter.
  - Recall the Taylor-expansion for a function $f(x)$, which is given by the following equation:

$$f(x + \Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \mathcal{O}(3)$$

  - Next, consider also the Taylor-series expansion of the natural logarithm of a variable x, i.e. $f(x) = ln(x)$. Ignoring all terms of degree three and higher, this Taylor-expansion is given by the following equation:

$$ln(x + \Delta x) \approx ln(x) + \frac{1}{x} \cdot \frac{1}{1!} \cdot \Delta x + \frac{-1}{x^2} \cdot \frac{1}{2!} \cdot \Delta x^2$$

$\Big|$ Provided that Δx is small already, we can safely ignore Δx^2.

$$\approx ln(x) + \frac{1}{x} \cdot \Delta x$$
$$\approx ln(x) + \frac{\Delta x}{x}$$

– We evaluate the function at the point where x is equal to one. Provided that x is small, the following statement is approximately true:

$$\begin{aligned} ln(1+\Delta x) &\approx ln(1) + \Delta x \\ &\approx \Delta x \end{aligned}$$

– The expression above has an important consequence, in a financial context. Id est it is possible to approximate simple returns with log returns when they are measured on a small time span.

13.1 Evolution of wealth in a riskless world

- The evolution of wealth in a riskless world is completely deterministic. There are no stochastic variables involved. We can model the evolution of wealth over time by compounding a capital sum over a certain time period, using a known interest rate r.

  – In the case of simple compounding, a capital sum S_{t_0} will capitalize over a time period τ to S_T. Mathematically:

  $$S_T = S_{t_0} \cdot (1 + r \cdot \tau)$$

  – In the case of periodic compounding, a capital sum S_{t_0} will capitalize over a time period τ to S_T. Mathematically:

  $$S_T = S_{t_0} \cdot (1 + \frac{r}{m})^{\tau \cdot m}$$

  – Continuous compounding is the limit case where the compounding frequency m frequency approaches infinity. Mathematically:

  $$\begin{aligned} \lim_{m\to\infty} (1 + \frac{r}{m})^{\tau \cdot m} &= \lim \lim_{m\to\infty} e^{ln((1+\frac{r}{m})^{\tau \cdot m})} \\ &= e^{(\tau \cdot m) \cdot ln(1+\frac{r}{m})} \\ &\approx e^{(\tau \cdot m) \cdot \frac{r}{m}} \\ &\approx e^{\tau \cdot r} \end{aligned}$$

- These models model the evolution of wealth in a deterministic way and will therefore only work in a riskless world. However, we are interested in how the value of a risky underlying asset evolves over time. To do this, we will have to model the value of the underlying as a stochastic variable.

13.2 Evolution of wealth in a risky world

- Consider the evolution of the value of a risky asset over time. The evolution of the value of such an asset is random in the long term. We therefore model the spot price of such an asset at a certain point in time as a stochastic variable. The value of this asset over time is therefore a series of stochastic variables, indexed by time. We say that the value of the underlying S behaves as a stochastic process. Mathematically, this becomes:

$$\{S_{t_0}, S_{t_1}, ..., S_T\}$$

- Notice that there are only realizations of the price of the asset at discrete points in time. Id est the time and the value of the underlying are modelled in a discrete manner.
- Suppose now that we let the time step Δt become infinitesimally small. Such a time step is denoted by dt. In that case, there are realizations for every infinitesimally small time step dt. Mathematically, this becomes:

$$\lim_{\Delta t \to 0} \Delta t = dt$$

- In this chapter, we will discuss different approaches to model the value of the risky asset in a continuous manner, these are called continuous time stochastic processes. These continuous time stochastic processes, include:
  - White noise
  - The Wiener process.
  - The generalized Wiener process.
  - The Ito process.

13.3 Gaussian white noise

- The first continuous time stochastic process we consider, is called Gaussian white noise. Suppose the value of an underlying asset is modelled by a stochastic variable Z. The value of the given asset at an arbitrary point in time is equal to Z_{t_i}. As stated before, the value of the asset is a series of stochastic variables, indexed by time. Mathematically, this becomes:

$$\{Z_{t_0}, Z_{t_1}, ..., Z_{t_i}, ...\}$$

- The value of the underlying after one period is computed as the sum of the value of the underlying in the previous period and a value, randomly drawn from a standard-normal distribution. Mathematically:

$$Z_{t_{i+1}} = Z_{t_i} + \epsilon_1$$

The random values, are independent and identical drawings from the same standard-normal distribution. Mathematically:

$$\epsilon_i \overset{idd}{\sim} \mathcal{N}(0,1)$$

The value of the underlying after two periods is therefore given by:

$$Z_{t_{i+2}} = Z_{t_i} + \epsilon_i + \epsilon_{i+1}$$

- We can easily expand these expressions for an arbitrary number of periods n:

$$\begin{aligned} Z_{t_0} &= Z_{t_0} + \epsilon_1 \\ Z_{t_2} &= Z_{t_0} + \epsilon_1 + \epsilon_2 \\ &... \\ Z_{t_n} &= Z_{t_0} + \epsilon_1 + \epsilon_2 ... + \epsilon_n \end{aligned}$$

- In general, the value of the asset, after an arbitrary number of periods i, is given by the following expression:

$$Z_{t_i} = Z_{t_0} + \sum_{j=1}^{i} \epsilon_j$$

Where $\epsilon_j \sim \mathcal{N}(0,1)$ are independent drawings from the same distribution.

- We now take a look at the properties of this process.

  1. Gaussian white noise is a stochastic process where we have a starting value Z_{t_0} and a series of realizations $Z_{t_0}, Z_{t_1}, ...$
  2. The value of the asset between one period and the next, is computed as:

$$\begin{aligned} Z_{t+1} &= Z_t + \epsilon_{t+1} \\ Z_{t+1} - Z_t &= \epsilon_t \\ &\equiv \Delta Z \end{aligned}$$

  3. The expected value of this ΔZ is equal to zero while the variance of ΔZ is equal to one. Mathematically, this becomes:

$$\begin{cases} E(\Delta Z) = 0 \\ \sigma^2(\Delta Z) = 1 \end{cases}$$

4. The difference in value of the asset, after n periods, is computed as the sum of n random independent drawings from the same standard-normal distribution. Mathematically, this becomes:

$$\begin{aligned} \Delta Z_n &= Z_{t+n} - Z_t \\ &= Z_t + \sum_{i=1}^{n} \epsilon_i - Z_t \\ &= \sum_{i=1}^{n} \epsilon_i \end{aligned}$$

5. The expectation of the difference in value of the asset after n periods is equal to zero. The variance of the difference in value of the asset after n periods is equal to the number of periods n; this property is called independent innovations.

$$\begin{aligned} E(\Delta Z_n) &= E(\sum_{i=1}^{n} \epsilon_i) \\ &= \sum_{i=1}^{n} E(\epsilon_i) \\ &= 0 \end{aligned}$$

$$\begin{aligned} var(\sum_{i=1}^{n} \epsilon_i) &= \sum_{i=1}^{n} var(\epsilon_i) \\ &= n \cdot 1 \\ &= n \end{aligned}$$

6. The expected value of the asset, after n periods is equal to the value of the asset at the present time, this property is characteristic for a Markov process. Id est this property is known as the Markov-property.

$$E(Z_{t+n}) = Z_t$$

13.4 The Wiener process

- Gaussian white noise has two important limitations. First of all, we want a process which has a built-in relationship with time. Id est the expected value and the variance of the stochastic process should vary with time. Such relationships is not present in Gaussian white noise. Furthermore, we cannot scale the volatility in Gaussian white noise. Id est we should be able to adapt the volatility of the process according to the idiosyncratic risk of a specific asset. We are thus confronted with two scaling problems.
- First, consider the problem of building in a relationship of the process with time. There are several possibilities that allow us to scale ΔZ with time.
  1. $\Delta Z = \epsilon \cdot \Delta t$
  2. $\Delta Z = \epsilon \cdot (\Delta t)^2$
  3. $\Delta Z = \epsilon \cdot \sqrt{\Delta t}$
- Consider the first candidate where we multiply the difference in the value of the asset in a given time period ϵ, with time t itself. Mathematically:

$$\Delta Z = \epsilon \cdot \Delta t$$

  We can easily see that the expectation of such a process is equal to zero while the variance is equal to the time period squared. Mathematically:

$$\begin{aligned} E(\Delta Z) &= E(\epsilon \cdot \Delta t) \\ &= \Delta t \cdot E(\epsilon) \\ &= 0 \end{aligned}$$

$$\begin{aligned} var(\Delta Z) &= var(\epsilon \cdot \Delta t) \\ &= (\Delta t)^2 \cdot var(\epsilon) \\ &= (\Delta t)^2 \cdot 1 \\ &= (\Delta t)^2 \end{aligned}$$

  For our purposes, we will choose Δt to be a small as possible. The variance therefore becomes negligibly small. If we want to go to a continuous time process, there is no point is scaling ϵ with Δt because the variance would become zero. The process would therefore come to a halt because both the variance and the expectation of the process would be zero.

- Consider the second candidate, where we multiply the difference in the value of the asset ϵ with the time period squared. Mathematically:

$$\Delta Z = \epsilon \cdot t^2$$

We can easily see that this second candidate has the same problem. Id est, the variance will become negligibly small, for small Δt. Mathematically:

$$\begin{aligned} var(\Delta Z) &= var(\epsilon \cdot (\Delta t)^2) \\ &= (\Delta t)^4 \cdot var(\epsilon) \\ &= (\Delta t)^4 \end{aligned}$$

- Consider the third candidate, where we multiply the difference in the value of the asset ϵ with the square root of the length of the time period. Mathematically:

$$\Delta Z = \epsilon \cdot \sqrt{\Delta t}$$

We can easily see that the expectation of this process is equal to zero, while the variance is equal to the time period itself. Mathematically:

$$\begin{aligned} E(\Delta Z) &= E(\epsilon \cdot \sqrt{t}) \\ &= \sqrt{\Delta t} \cdot E(\epsilon) \\ &= 0 \end{aligned}$$

$$\begin{aligned} var(\Delta Z) &= var(\epsilon \cdot \sqrt{\Delta t}) \\ &= \Delta t \cdot var(\epsilon) \\ &= \Delta t \end{aligned}$$

- In conclusion, if we want to scale the process with time, it is better to use the square root of time. In doing so, we make the variance proportional with time and we avoid that the variance becomes negligible. We therefore adapt our model by multiplying the difference in the value of the asset over a given time period ϵ, with the square root of the time period $\sqrt{\Delta t}$. Mathematically:

$$\Delta Z = \epsilon \cdot \sqrt{\Delta t}$$

- We will now examine this process over multiple time periods. Suppose there are n time steps i.e. the time to maturity is divided into n different periods. The size of such a time step is given by:

$$\Delta t = \frac{\tau}{n}$$

- We are interested in the difference in the value of the asset between time t_0 and time T. This difference is of course equal to the sum of the differences in the value between each subsequent period. Mathematically:

$$\begin{aligned} Z_T - Z_{t_0} &= \sum_{i=1}^{n} \Delta Z_i \\ &= \sum_{i=1}^{n} \epsilon_i \cdot \sqrt{\Delta t} \end{aligned}$$

- The expected value of the difference in the value of the asset, between time t_0 is equal to zero. Mathematically:

$$\begin{aligned} E(Z_T - Z_{t_0}) &= E(Z_T) - E(Z_{t_0}) \\ &= 0 \end{aligned}$$

- The variance of the difference in the value of the asset, between time T and time t_0 is equal to the length of the entire period. Mathematically:

$$\begin{aligned} var(Z_T - Z_{t_0}) &= var(\sum_{i=1}^{n} \epsilon_i \cdot \sqrt{\Delta t}) \\ &= \Delta t \cdot var(\sum_{i=1}^{n} \epsilon_i) \\ &= n \cdot \Delta t \\ &= (T - t_0) \\ &\equiv \tau \end{aligned}$$

- Suppose we let the time interval Δt become infinitesimally small. The expression that we derive for this limiting case characterizes the Wiener process which is also known as the standard Brownian motion. Mathematically:

$$\begin{aligned} \lim_{\Delta t \to dt} (\Delta Z) &= \lim_{\Delta t \to dt} \epsilon \cdot \sqrt{\Delta t} \\ dz &= \epsilon \cdot \sqrt{dt} \end{aligned}$$

13.5 The generalized Wiener process.

- In the standard Brownian motion, price changes in the underlying asset, during a given time period, are proportional to that time period. This reflects greater price changes over longer periods of time.
- However, this process will still revolve around zero due to the fact that the expectation of the process is equal to zero. Id est there is no drift. A process that reflects the price behaviour of an asset needs to incorporate a drift for in normal circumstances, asset prices increase over time.
- Our goal is to incorporate a drift in the process that is proportional to time. Id est the value of the asset should increase with time. Mathematically, this drift becomes:

$$a \cdot \Delta t$$

- In doing so, we have built in a linear trend in the model. Our updated model consists of two terms. The first term reflects the trend in the price of the asset whereas the second term reflects the volatility in the price of the asset. Note that the second term is stochastic in nature while the first term is completely deterministic. Mathematically:

$$dx = a \cdot dt + dz$$

- We now address the second problem we discussed with Gaussian white noise; being that it is not possible to scale the volatility of the process. In the case of the standard Brownian motion, the volatility of the process was equal to the length of the time period. The volatility therefore increases with time. This reflected the fact that, the further we go in time, the less certainty there exists about the value of the asset. We can multiply the term that is responsible for the volatility dz with a factor b to control for the size of the volatility. Mathematically:

$$b \cdot dz$$

- We have scaled the volatility of the process. Our updated model again consists of two terms. The first term still reflects the trend in the price of the asset whereas the second term still reflects the volatility in the price of the asset. This process is known as the arithmetic Brownian motion. The term $a \cdot dt$ is called the instantaneous drift and the term $b \cdot dz$ is called the instantaneous volatility. Mathematically, this becomes:

$$dx = a \cdot dt + b \cdot dz$$

- Note that for the case where a is equal to zero and b is equal to one; the arithmetic Brownian motion becomes a standard Brownian motion. Id est the standard Brownian motion is nested inside the geometric Brownian motion. Mathematically:

$$\begin{aligned} dx &= a \cdot dt + b \cdot dz \\ &= 0 \cdot dt + 1 \cdot dz \\ &= dz \end{aligned}$$

- We can summarize all of the above, with the following set of equations:

$$\begin{cases} Z \sim \mathcal{N}(0, 1) \\ X = Z \cdot \sigma + \mu \\ dx = a \cdot dt + b \cdot dz \end{cases}$$

- We now take a closer look at the properties of the arithmetic Brownian motion. As we already know, the arithmetic Brownian motion contains both a deterministic term which is responsible for the drift of the process and a stochastic term which is responsible for the volatility of the process. Mathematically:

$$dx = a \cdot dt + b \cdot dz$$

  - The stochastic term is of no importance for the expectation of the process. Mathematically:

$$\begin{aligned} E(dx) &= E(adt + bdz) \\ &= E(adt) + E(bdz) \\ &= adt + b \cdot E(dz) \\ &= adt + b \cdot E(\epsilon\sqrt{dt}) \\ &= adt + b \cdot \sqrt{dt} \cdot E(\epsilon) \\ &= adt \end{aligned}$$

  - The deterministic term is on its turn of no importance for the variance or the volatility of the process. Notice also that the variance is proportional to the time step dt. Mathematically:

$$\begin{aligned} var(dx) &= var(adt + bdz) \\ &= var(bdz) \\ &= b^2 \cdot var(\sqrt{dt} \cdot \epsilon) \\ &= b^2 \cdot dt \cdot var(\epsilon) \\ &= b^2 \cdot dt \end{aligned}$$

– We now consider the difference in the value of the asset over a time period of length $x_T - x_{t_0}$, where there are n time steps. Mathematically:

$$\Delta t = \frac{\tau}{n}$$

– The expectation of this price difference is equal to a, multiplied by the length of the time period.

$$\begin{aligned}
E(x_T - x_{t_0}) &= E(\sum_{i=1}^{n} \Delta x) \\
&= E(\sum_{i=1}^{n} a\Delta t + b \cdot \epsilon \cdot \sqrt{\Delta t}) \\
&= E(\sum_{i=1}^{n} a\Delta t) + E(\sum_{i=1}^{n} b \cdot \epsilon \cdot \sqrt{\Delta t}) \\
&= a \cdot n\Delta t + n \cdot b \cdot \sqrt{\Delta t} \cdot E(\epsilon) \\
&= a \cdot \tau + 0 \\
&= a\tau
\end{aligned}$$

– The variance of this difference is equal to b, multiplied by the length of the time period.

$$\begin{aligned}
var(x_T - x_{t_0}) &= var(\sum_{i=1}^{n} a \cdot \Delta t + b \cdot \sqrt{\Delta t} \cdot \epsilon) \\
&= var(\sum_{i=1}^{n} b \cdot \sqrt{\Delta t} \cdot \epsilon) \\
&= \sum_{i=1}^{n} var(b \cdot \sqrt{\Delta t} \cdot \epsilon) \\
&= \sum_{i=1}^{n} b^2 \cdot \Delta t \cdot var(\epsilon) \\
&= b^2 \cdot \Delta t \cdot \sum_{i=1}^{n} var(\epsilon) \\
&= b^2 \cdot \Delta t \cdot n \\
&= b^2\tau
\end{aligned}$$

- In summary, dz is the standard Brownian motion. In this process, there is no drift and the volatility is proportional to time. The geometric Brownian motion incorporates a drift $(a \cdot dt)$ into the process, which is proportional to time. The geometric Brownian motion also scales the volatility of the process. This corresponds with stretching the process with time.

- Positive and negative cashflows are possible within this model. For this reason, this process is not suitable to model stock prices. Furthermore, this model represents linear growth while stock prices tend to grow in an exponential manner.

13.6 The Ito process

- By now, we have derived the expression for the arithmetic Brownian motion. In the case of the arithmetic Brownian motion, the trend is linear. However, in general, stock prices tend to grow in an exponential manner. Recall that in the case of exponential growth, relative changes remain equal. We can easily adapt the arithmetic Brownian motion to reflect exponential growth by adapting the model to reflect relative price changes instead of absolute price changes. The relative change in the price of an underlying asset then grows with a certain trend udt and is subject to a volatility of σdz. Mathematically, this becomes:

$$\frac{dS}{S} = u \cdot dt + \sigma \cdot dz$$

- This process is an Ito-process or more specifically, the geometric Brownian motion. Notice that both the drift term and the diffusion term are proportional to the price of the underlying S. Id est, both are a function of S. This implies that both the linear trend and the volatility scale up with the height of the price of the underlying asset. Mathematically, this becomes:

$$ds = u \cdot S \cdot dt + \sigma \cdot S \cdot dz$$

- In summary, the Ito process is the arithmetic Brownian motion where we make the drift term and the volatility a function of x. The geometric Brownian motion is a specific case of the Ito-process where the drift term $a(x,t) \cdot dt$ is equal to $ux \cdot dt$ and where the volatility $b(x,t) \cdot dt$ is equal to $\sigma x \cdot dt$. Mathematically:

$$dx = a(x,t) \cdot dt + b(x,t) \cdot dz$$
$$dx = ux \cdot dt + \sigma x \cdot dz$$

13.7 The Ornstein-Uhlenbeck process

- In some applications, the variable x follows a mean-reverting process. The coefficient in the drift term is then no longer fixed but is dependent on the distance between the actual value and the mean value of the independent variable. Mathematically, such a process can be expressed by the following equation:

$$dx = k \cdot (\mu - x) \cdot dt + \sigma \cdot t^{\gamma}$$

- The first term is the drift term which is now dependent on the distance between the actual value and the mean value of the variable x. The coefficient k reflects the speed of the mean reversion. In the long term, the process returns to its original level, hence its name.
- This process can be used to model an interest rate. Under normal circumstances, interest rates are not expected to behave as a geometric Brownian motion in the long term because an interest rate returns to its original levels in the long term.

13.8 Discretization

- Consider the geometric Brownian motion, which is a continuous time process. The discretized version of the geometric Brownian motion does not consider instantaneous price changes dS and instantaneous time changes dt but rather changes in price ΔS and in time Δt i.e. changes over a certain price range and over a certain time period. Mathematically:

$$\frac{\Delta S}{S_t} = \mu \cdot \Delta t + \sigma \cdot \sqrt{\Delta t} \cdot \epsilon$$

- The expression above defines the relative price change of an underlying asset between two points in time t and $t+1$. Ignoring the volatility, the price of the underlying asset at time $t+1$ is therefore equal to the price of the underlying asset at time t multiplied with this relative price change $\frac{\Delta S}{S_t}$. Mathematically, this becomes:

$$S_{t+1} = S_t \cdot (1 + \frac{\Delta S}{S_t})$$

- Consider the limit case where the time step Δt becomes infinitesimally small. Mathematically:

$$\lim_{\Delta t \to dt} \left(S_{t+1}\right) = \lim_{\Delta t \to dt} \left(S_t \cdot (1 + \frac{\Delta S}{S_t})\right)$$
$$S_{t+1} = S_t \cdot e^{u \cdot dt}$$

- We now do consider the volatility in the price of the underlying. This corresponds to increasing the exponent with σdz.

$$S_{t+1} = S_t \cdot e^{(udt + \sigma dz)}$$

- This discretized version of the geometric Brownian motion process models the price of the underlying as revolving around an exponentially increasing function with time. The discretized version of this process is given by the equation below.

$$S_{t_1} \approx S_t \cdot e^{u\Delta t + \sigma\sqrt{\delta t} \cdot \epsilon}$$

13.9 Working with random functions

- The derivative of an arbitrary function $f(x)$, is given by:

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- The derivative gives the rate of change and measures the impact of one variable on another variable. In essence, we give a small shock to x, measure the change in the function value and divide by the size of the shock.
- Suppose now that we want to compute the derivative of a stochastic variable. Suppose for example that x follows a standard Brownian motion. Mathematically:

$$dx = \epsilon \cdot \sqrt{dt}$$

- Suppose we build a function of x, for example x^2 Mathematically:

$$f(x) = x^2$$

  Because x is a stochastic variable, we cannot easily derive the compute the derivative of this function.

13.10 The Ito-Doeblin Lemma

- The Ito-Doeblin lemma states that, if a variable x follows an Ito-process, then a specific function $G(x)$, also follows an Ito-process.
- This function $G(x)$ consists of a deterministic part and a stochastic part. The uncertainty of both x and $G(x)$ depends on the same stochastic variable dz. Therefore, if we have a model for the value of an underlying asset x at our disposal; the function of the value of the underlying x i.e. $G(x)$ will depend on the same source of uncertainty as the value of the underlying asset x. Notice that the value of a derivative contract is always a function of the value of the underlying asset. Mathematically:
  - Suppose that x follows an Ito process:

$$dx = a(x,t) \cdot dt + b(x,t) \cdot dz$$

  - In that case, $G(x)$ also has to follow an Ito process:

$$dG = (G_x a + G_t + 0.5 G_{xx} \cdot b^2) \cdot dt + (G_x \cdot b) \cdot dz$$

- We will not provide a real proof of Ito's lemma in this section. However, we will provide a proof based on an heuristic approach.
  - First, assume that x follows an Ito-process and let G be a continuous and differentiable function of x and t. Mathematically:

$$dx = a(x,t) \cdot dt + b(x,t) \cdot dz$$

  - In the first step, we discretize the expression for the Ito process: Mathematically:

$$\Delta x = a(x,t) \cdot \Delta t + b(x,t) \cdot \epsilon \cdot \sqrt{\Delta t}$$

  - We derive the expression for Δx^2 and for $\Delta x \Delta t$, in the limiting case, where $\Delta t \to 0$ and $\Delta S \to 0$.

$$\begin{aligned}
\Delta x^2 &= \left[a(x,t) \cdot \Delta t + b(x,t) \cdot \epsilon \cdot \sqrt{\Delta t}\right]^2 \\
&= a(x,t)^2 \cdot \Delta t^2 + b(x,t)^2 \cdot \epsilon^2 \cdot \Delta t + 2 \cdot a(x,t) \cdot b(x,t) \cdot \epsilon \cdot \Delta t^{\frac{3}{2}} \\
&= b(x,t)^2 \cdot \epsilon^2 \cdot \Delta t
\end{aligned}$$

$$\begin{aligned}
\Delta x \Delta t &= \left[a(x,t) \cdot \Delta t + b(x,t) \cdot \sqrt{\Delta t}\right] \cdot \Delta t \\
&= a(x,t) \cdot (\Delta t)^2 + b(x,t) \cdot (\Delta t)^{\frac{3}{2}} \\
&= 0
\end{aligned}$$

- Consider what happens to $\epsilon^2 \cdot \Delta t$. If Δt becomes very small, the variance becomes zero. There is no more variance left. In that case, $\epsilon^2 \cdot \Delta t$ becomes Δt. Mathematically:

$$\begin{aligned} E(\epsilon^2 \cdot \Delta t) &= \Delta t \cdot E(\epsilon^2) \\ &= \Delta t \end{aligned}$$

$$\begin{aligned} var(\epsilon^2 \cdot \Delta t) &= E(\epsilon^2 \cdot \Delta t - \Delta t)^2 \\ &= 0 \end{aligned}$$

- With this, we can modify the expression for Δx^2. Mathematically:

$$\begin{aligned} \Delta x^2 &= b(x,t)^2 \cdot \epsilon^2 \cdot \Delta t \\ &= b(x,t)^2 \cdot \Delta t \end{aligned}$$

- Consider also the Taylor-series expansion of the 2$^{\text{nd}}$ order for the function G. Mathematically:

$$\begin{aligned} \Delta G &= \frac{\delta G}{\delta x} \cdot \Delta x + \frac{\delta G}{\delta t} \cdot \Delta t + \frac{1}{2} \cdot \frac{\delta^2 G}{\delta x^2} \cdot \Delta x^2 + \frac{1}{2} \cdot \frac{\delta^2 G}{\delta t^2} \cdot \Delta t^2 + \frac{1}{2} \cdot \frac{\delta^2 G}{\delta x \delta t} \cdot \Delta x \cdot \Delta t + \mathcal{O}(3) \\ &= \frac{\delta G}{\delta x} \cdot \Delta x + \frac{\delta G}{\delta t} \cdot \Delta t + \frac{1}{2} \cdot \frac{\delta^2 G}{\delta x^2} \cdot b(x,t)^2 \cdot \Delta t + \mathcal{O}(3) \\ &= \frac{\delta G}{\delta x} \cdot \Delta x + \frac{\delta G}{\delta t} \cdot \Delta t + \frac{1}{2} \cdot \frac{\delta^2 G}{\delta x^2} \cdot b(x,t)^2 \cdot \Delta t \end{aligned}$$

$$dG = \frac{\delta G}{\delta x} \cdot dx + \frac{\delta G}{\delta t} \cdot dt + \frac{1}{2} \cdot \frac{\delta^2 G}{\delta x^2} \cdot b(x,t)^2 \cdot dt$$

- We can substitute dx into this equation for the Ito process. Mathematically:

$$\begin{aligned} dG &= \frac{\delta G}{\delta x} \Big[a(x,t) \cdot dt + b(x,t) \cdot dz \Big] + \frac{\delta G}{\delta t} \cdot dt + \frac{1}{2} \cdot \frac{\delta^2 G}{\delta x^2} \cdot b(x,t)^2 \cdot dt \\ &= \Big[\frac{\delta G}{\delta x} \cdot a(x,t) + \frac{\delta G}{\delta t} + \frac{1}{2} \cdot \frac{\delta^2 G}{\delta x^2} \cdot b(x,t)^2 \Big] \cdot dt + \frac{\delta G}{\delta x} \cdot b(x,t) \cdot dz \end{aligned}$$

13.11 Financial applications

- First, consider a forward contract on an underlying that provides no income during the lifespan of the forward contract.
  - The forward price is defined as:

$$F = S_{t_0} \cdot e^{r(T-t_0)}$$

  - Suppose the price of the underlying asset S follows a geometric Brownian motion:

$$dS = \mu \cdot S \cdot dt + \sigma \cdot S \cdot dz$$

  - Our goal is to derive the stochastic process for the forward price F. To do this, we first compute the first derivatives of the forward price with regard to time, of the forward price with regard to the price of the underlying and the second derivative of the forward price with regard to the price of the underlying.

$$\frac{\delta F}{\delta S} = e^{r\cdot(T-t)}$$
$$\frac{\delta F}{\delta t} = -r \cdot S \cdot e^{r\cdot(T-t)}$$
$$\frac{\delta^2 F}{\delta S^2} = 0$$

  - Next, we determine the value of a and b:

$$a = \mu \cdot S$$
$$b = \sigma \cdot S$$

  - Finally, we substitute the results in Ito's lemma:

$$\begin{aligned} dF &= \left[e^{r\cdot(T-t)} \cdot \mu \cdot S - r \cdot S \cdot e^{r\cdot(T-t)}\right] \cdot dt + e^{r(T-t)} \cdot \sigma \cdot S \cdot dz \\ &= \left[(\mu - r) \cdot F\right] \cdot dt + \sigma \cdot F \cdot dz \end{aligned}$$

- Suppose we were to model the price of an underlying asset with the geometric Brownian motion. This would imply that the price of the underlying asset could becomes negative which is of course impossible. For this reason, we take the function $G(S)$ to be the natural logarithm of the price of the underlying asset. In doing so, we avoid prices becoming negative.
  - The function $G(S)$ thus represents the natural logarithm of the price of the underlying. Mathematically:
  $$G(S) = ln(S)$$
  - The expression for dG represents the change in the log prices after an infinitesimally small time period dt. This price change is equivalent with the logarithm of the gross return in this time period. Mathematically:
  $$ln(S_{t+dt}) - ln(S_t) = ln(\frac{S_{t+dt}}{S_t})$$
  - In the first step, we compute the derivatives:
  $$\frac{\delta G}{\delta S} = \frac{1}{S}$$
  $$\frac{\delta^2 G}{\delta S^2} = -\frac{1}{S^2}$$
  $$\frac{\delta G}{\delta t} = 0$$
  - In the second step, we determine the value for a and b:
  $$a = \mu \cdot S$$
  $$b = \sigma \cdot S$$
  - In the last step, we substitute these findings in Ito's lemma. We then find an expression for the log-returns dG. Mathematically:
  $$\begin{aligned} dG &= \left[\frac{1}{S} \cdot \mu \cdot S + \frac{1}{2} \cdot (-\frac{1}{S^2}) \cdot (\sigma \cdot S)^2\right] \cdot dt + \frac{1}{S} \cdot \sigma \cdot S \cdot dz \\ &= (\mu - \frac{1}{2} \cdot \sigma^2) \cdot dt + \sigma \cdot dz \end{aligned}$$
  - We conclude that $G(S)$ follows a generalized Wiener process i.e. the drift is constant and the volatility is constant. The instantaneous volatility is the same for the relative price changes of the underlying as for the price changes of the derivative. The instantaneous drift for the price of the derivative is equal to the instantaneous drift the relative price changes of the underlying asset, plus half of the variance.

– Notice that the log-returns dG are normally distributed because the only stochastic element was dz which itself was normally distributed.

$$dG = ln(S_{t+dt}) - ln(S_t) \sim \mathcal{N}\Big((\mu - \frac{1}{2} \cdot \sigma^2) \cdot dt, \sigma \cdot \sqrt{dt}\Big)$$

– This implies that the prices themselves are log-normally distributed. Mathematically:

$$\begin{aligned} ln(S_{t+dt}) &\sim \mathcal{N}\Big[ln(S_t) + (\mu - \frac{1}{2} \cdot \sigma^2) \cdot dt, \sigma \cdot \sqrt{dt}\Big] \\ S_{t+dt} &\sim \mathcal{LN} \end{aligned}$$

– In summary, when the log-returns of the underlying are conjectured to follow a normal distribution, the prices of the underlying follow a log-normal distribution. This implies that returns can take on any real value, but prices, as modelled by the function $G(S)$, cannot become negative. In option pricing we therefore favour log-returns. Mathematically:

$$\begin{aligned} &dG \in [-\infty, +\infty] \\ &S \cdot e^{-\infty} = 0 \end{aligned}$$

13.12 The log-normal distribution

- Assume that the returns over a given period, starting at time t_0 and ending at time T, are normally distributed with mean $\mu_c T$ and with variance $\sigma^2 T$. Mathematically, this becomes:

$$\begin{aligned} E\Big[\frac{S_T}{S_{t_0}}\Big] &= \mu_c T \\ var\Big[\frac{S_T}{S_{t_0}}\Big] &= \sigma^2 T \end{aligned}$$

- Assume now that the returns over the given period are log-normally distributed. The log-normal distribution assumes that the natural logarithm of the returns are normally distributed:

$$ln(\frac{S_{t_1}}{S_{t_0}}) \sim \mathcal{N}(\mu_c, \sigma)$$

- The standard properties of the log-normal distribution tell us that:

$$\begin{aligned} E\Big[\frac{S_T}{S_{t_0}}\Big] &= e^{\mu_c \tau + 0.5\sigma^2 \tau} \\ var(\frac{S_T}{S_{t_0}}) &= e^{2\mu_c \tau + 2\sigma^2 \tau} - e^{2\mu_c \tau + \sigma^2 \tau} \end{aligned}$$

- This implies that, in the case of a risk-free asset, the returns are constant. Mathematically:

$$\frac{S_T}{S_{t_0}} = e^{\mu_c T}$$
$$\mu_c T = ln(\frac{S_T}{S_{t_0}})$$

13.13 Confidence intervals

- Suppose the returns of a stock, with a current price which is equal to \$100, follow a normal distribution. The mean of these returns is equal to 10% while the variance is equal to 30%. We are interested in the 95% confidence interval for the stock price, in three months time.
  - First we calculate the mean and variance for the gross returns within three months time. Mathematically, this becomes:

$$\begin{aligned} \mu_c T &= 0.10 \cdot 0.25 \\ &= 0.025 \\ \sigma^2 T &= 0.3^2 \cdot 0.25 \\ &= 0.0225 \end{aligned}$$

  - The gross returns within three months time are thus normally distributed with mean 0.025 and standard deviation 0.0225. Mathematically:

$$ln(\frac{S_T}{100}) \sim \mathcal{N}(0.025, 0.0225)$$

  -
  - We can construct a 95% confidence interval for the gross returns of the stock. From this we can easily derive the 95% confidence interval for the price of the stock at maturity. Mathematically, this becomes:

$$\begin{aligned} &P(0.025 + 1.96 \cdot 0.0225 < 0.025 - 1.96 \cdot 0.0225 < ln(\frac{S_T}{100})) \\ &= P(-0.269 < ln(\frac{S_T}{100}) < 0.319) \\ &= P(e^{-0.269} < e^{ln(\frac{S_T}{100})} < e^{0.319}) \\ &= P(76.41 < S_T < 137.58) \end{aligned}$$

14 The Black-Scholes-Merton model

14.1 Assumptions

- In this chapter, we adhere to the following assumptions:
  - Stocks are continuously traded.
  - Stocks follow a geometric Brownian motion which implies that the volatility is constant.
  - Short selling is allowed.
  - There are no transaction costs.
  - Assets are perfectly divisible.
  - There are no dividends during the lifetime of option contracts.
  - The riskless interest rate is constant and the term structure of interest rates is flat.

14.2 Introduction

- Recall Ito's lemma. If a variable x follows an Ito-process, then the function $G(x)$ of this variable also follows an Ito-process. Mathematically, this becomes

$$\begin{aligned} dx &= a(x,t)dt + b(x,t)dz \\ dG &= (G_x \cdot a + G_t + 0.5 \cdot G_{xx} \cdot b^2) \cdot dt + (G_x \cdot b) \cdot dz \end{aligned}$$

- Suppose we construct a portfolio, containing:
  1. One unit in a short derivative for which the value is given by G.
  2. G_x units of the asset itself. Where the value of this asset is given by x.
- The value of this portfolio Π can therefore be computed in the following manner:

$$\Pi = -G + G_x \cdot x$$

- We can then derive an expression for a change in the value of the portfolio $d\Pi$ by substituting the expressions for dx and dG, in the expression for the value for the portfolio. The resulting expression is completely deterministic i.e. there is no stochastic term left in the equation. This implies that the portfolio is a riskless investment. The portfolio should therefore yield the riskless interest rate.

$$\begin{aligned} d\Pi &= -dG + G_x dx \\ &= -(G_x \cdot a + G_t + 0.5 \cdot G_{xx} \cdot b^2) \cdot dt - (G_x \cdot b) \cdot dz \\ &= -(G_t + 0.5 \cdot G_{xx} \cdot b^2) \cdot dt \end{aligned}$$

- A riskless position of size Π will earn interest $r \cdot \Pi \cdot dt$ over an infinitesimally small time perioddt. Mathematically, this becomes:

$$d\Pi = r \cdot \Pi \cdot dt$$

- Substituting the expressions for Π and $d\Pi$ in the equation above yields a partial differential equation.

$$\begin{aligned} d\Pi &= r \cdot \Pi \cdot dt \\ -\Big[G_t + 0.5 \cdot G_{xx} \cdot b^2\Big] \cdot dt &= r \cdot \Big[-G + G_x \cdot x\Big] \cdot dt \\ -G_t - 0.5 \cdot G_{xx} \cdot b^2 &= -r \cdot G + r \cdot G_x \cdot x \\ G_t + r \cdot G_x \cdot x + 0.5 \cdot G_{xx} \cdot b^2 &= r \cdot G \end{aligned}$$

- To solve this partial differential equation, we have to look for functions that satisfy the partial differential equation. Solving a partial differential equation does not yield a unique solution. All financial derivatives share the same partial differential equation, but they differ in their boundary condition. Additional information is required in order to determine a unique solution. This information entails the value of the payoff at maturity. Id est the boundary condition for a specific financial derivative is the payoff function for that financial derivative, at maturity.
- Any function $f(s,t)$ that is a solution to the partial differential equation, gives the theoretical price of the derivative in question. Any function that is not a solution to the partial differential equation, yields prices for the derivative in question, for which arbitrage is possible.

14.3 Risk-neutral valuation

- To solve the partial differential equation, we use the Feynman-Kac theorem which restates the equation in a different way. We assumed that the underlying asset yields the risk-free interest rate. In order to compute the price of a given derivative, we will have to compute the expected payoff from the given derivative and discount this expected payoff at the risk-free rate, back to the present.
- First, we consider a long forward contract.
  - We assume the underlying asset yields the risk-free interest rate.
  - We compute the expected cashflow from the payoff of the forward contract.

$$f_T^{Long\ fw.} = S_T \cdot e^{(r \cdot \tau)} - K$$

  - We compute the present value of the expected cashflow using the risk-free interest rate.

$$\begin{aligned} f_{t_0}^{Long\ fw.} &= (S_T \cdot e^{(r \cdot \tau)} - K) \cdot e^{(-r \cdot \tau)} \\ &= S_T - K \cdot e^{(-r \cdot \tau)} \end{aligned}$$

- Next, consider a European long call option.
  - First, assume that the underlying asset yield the risk-free interest rate and that the price of the underlying follows a geometric Brownian motion. Mathematically, this becomes:

$$\begin{aligned} dS &= r \cdot S \cdot dt + \sigma \cdot S \cdot dz \\ ln(S_T) &\sim \mathcal{N}\Big(ln(S_{t_0} + (r - \frac{1}{2} \cdot \sigma^2) \cdot \tau, \sigma \cdot \sqrt{\tau}\Big) \\ S_T &= S_{t_0} \cdot e^{(r - \frac{1}{2} \cdot \sigma^2) \cdot \tau + \sigma \cdot \sqrt{\tau} \cdot \epsilon} \end{aligned}$$

  - Next, we compute the expected cashflow of a European call option from the payoff at maturity.

$$\begin{aligned} E(f_T^{Eur.\ long\ call}) &= E(max\ (S_T - K, 0)) \\ &= \int_{-\infty}^{+\infty} [max(S_T - K, 0) \cdot \phi] \cdot d\epsilon \end{aligned}$$

- The expression for the expectation of the payoff of the option contains a term with a maximum operator. This term represents the payoff of the option, which is of course dependent on the value of the underlying at maturity. Our goal is to get rid of this maximum operator. Notice that the the payoff should always be greater than or equal to zero. The payoff will only be greater than or equal to zero in the case where the price of the underlying at maturity S_T is greater than or equal to the strike price K. Mathematically, this becomes:

$$S_T \geq K$$

- Notice also that the expression for the expectation of the payoff of the option was given by an integral over ϵ. We are therefore interested in rewriting the inequality above to express for what values of ϵ the inequality holds true. Mathematically, this becomes:

$$\begin{aligned} S_{t_0} \cdot e^{(r-\frac{1}{2}\cdot\sigma^2)\cdot\tau+\sigma\cdot\sqrt{\tau}\cdot\epsilon^*} &\geq K \\ (r - \frac{1}{2} \cdot \sigma^2) \cdot \tau + \sigma \cdot \sqrt{\tau} \cdot \epsilon^* &\geq ln(\frac{K}{S_{t_0}}) \\ \epsilon^* &\geq \frac{ln(\frac{K}{S_{t_0}}) - (r - \frac{1}{2} \cdot \sigma^2) \cdot \tau}{\sigma \cdot \sqrt{\tau}} \end{aligned}$$

- By adjusting the lower bound of the integral, we take into account only those cases for which the payoff is either positive or equal to zero. In adjusting the lower bound of the integral, the maximum operator became irrelevant. Mathematically, this becomes:

$$E(f_T^{Eur.\ long\ call}) = \int_{\epsilon^*}^{+\infty} [S_T - K] \cdot [\frac{1}{\sqrt{2\cdot\pi}} \cdot e^{\frac{-1}{2}\cdot\epsilon^2}] \cdot d\epsilon$$

- We can furthermore substitute the expression we found earlier for the value of the underlying asset at the maturity S_T. After this, we split the integral. Mathematically, this becomes:

$$\begin{aligned} E(f_T^{Eur.\ long\ call}) &= \int_{\epsilon^*}^{+\infty} [S_{t_0} \cdot e^{(r-\frac{1}{2}\cdot\sigma^2)\cdot\tau+\sigma\cdot\sqrt{\tau}\cdot\epsilon} - K] \cdot [\frac{1}{\sqrt{2\cdot\pi}} \cdot e^{\frac{-1}{2}\cdot\epsilon^2}] \cdot d\epsilon \\ &= \int_{\epsilon^*}^{+\infty} [S_{t_0} \cdot e^{(r-\frac{1}{2}\cdot\sigma^2)\cdot\tau+\sigma\cdot\sqrt{\tau}\cdot\epsilon}] - K \cdot \int_{\epsilon^*}^{+\infty} \frac{1}{\sqrt{2\cdot\pi}} \cdot e^{\frac{-1}{2}\cdot\epsilon^2} \cdot d\epsilon \end{aligned}$$

- Notice that part of the integral corresponds to the expression of a normal distribution. Mathematically:

$$\begin{aligned} \int_{\epsilon^*}^{+\infty} \frac{1}{\sqrt{2\cdot\pi}} \cdot d\epsilon &= P(\epsilon \geq \epsilon^*) \\ &= P(\epsilon \leq -\epsilon^*) \\ &= \mathcal{N}(-\epsilon^*) \end{aligned}$$

– Taking this fact into account, we can rewrite the expression for the expectation of the value of the European-style long call option, at maturity. Mathematically, this becomes:

$$\begin{aligned} E(f_T^{Eur.\ long\ call}) &= \int_{\epsilon^*}^{+\infty} S_{t_0} \cdot e^{(r-\frac{1}{2}\cdot\sigma^2)\cdot\tau+\sigma\cdot\sqrt{\tau}\cdot\epsilon} \cdot \frac{1}{\sqrt{2\cdot\pi}} \cdot e^{-\frac{1}{2}\cdot\epsilon^2} \cdot d\epsilon - K \cdot \mathcal{N}(\epsilon^*) \\ &= \int_{\epsilon^*}^{+\infty} S_{t_0} \cdot e^{(r-\frac{1}{2}\cdot\sigma^2)\cdot\tau+\sigma\cdot\sqrt{\tau}\cdot\epsilon} \cdot \frac{1}{\sqrt{2\cdot\pi}} \cdot e^{\sigma\cdot\sqrt{\tau}\cdot\epsilon-\frac{1}{2}\cdot\epsilon^2} \cdot d\epsilon - K \cdot \mathcal{N}(-\epsilon^*) \end{aligned}$$

– Notice the the following holds true:

$$\sigma \cdot \sqrt{T} - \frac{1}{2} \cdot \epsilon^2 = -\frac{1}{2} \cdot (\epsilon - \sigma \cdot \sqrt{T})^2 + \frac{1}{2} \cdot \sigma^2 \cdot T$$

– With this, we can again rewrite the equation for the expectation of the value of the European-style long call option, at maturity.

$$\begin{aligned} E(f_T^{Eur.\ long\ call}) &= S_{t_0} \cdot e^{r-\frac{1}{2}\cdot\sigma^2\cdot\tau} \cdot \int_{\epsilon^*}^{+\infty} \frac{1}{\sqrt{2\cdot\pi}} \cdot e^{-\frac{1}{2}\cdot(\epsilon-\sigma\cdot\sqrt{\tau})^2+\frac{1}{2}\cdot\sigma^2\cdot\tau} \cdot d\epsilon \\ &= S_{t_0} \cdot e^{r-\frac{1}{2}\cdot\sigma^2\cdot\tau+\frac{1}{2}\cdot\sigma^2\cdot\tau} \cdot \int_{\epsilon^*}^{+\infty} \frac{1}{\sqrt{2\cdot\pi}} \cdot e^{(-\frac{1}{2})\cdot(\epsilon-\sigma\cdot\sqrt{\tau})^2} \cdot d\epsilon - K \cdot \mathcal{N}(\epsilon^*) \\ &= S_{t_0} \cdot e^{r\cdot\tau} \cdot \int_{\epsilon^*}^{+\infty} \frac{1}{\sqrt{2\cdot\pi}} \cdot e^{-\frac{1}{2}\cdot(\epsilon-\sigma\cdot\sqrt{\tau})^2} \cdot d\epsilon - K \cdot \mathcal{N}(\epsilon^*) \\ &= S_{t_0} \cdot e^{r\cdot\tau} \cdot \int_{\epsilon^*}^{+\infty} \frac{1}{\sqrt{2\cdot\pi}} \cdot e^{-\frac{1}{2}\cdot(\epsilon-\sigma\cdot\sqrt{\tau})^2} \cdot d\epsilon - K \cdot \mathcal{N}(-\epsilon^*) \end{aligned}$$

– Notice that the integral corresponds to and can be written in terms of a probability. Mathematically, this becomes:

$$\begin{aligned} \int_{\epsilon^*}^{+\infty} \frac{1}{\sqrt{2\cdot\pi}} \cdot e^{-\frac{1}{2}\cdot(\epsilon-\sigma\cdot\sqrt{\tau})^2} \cdot d\epsilon &= P(\epsilon \geq \epsilon^*) \\ &= P(\epsilon - \sigma \cdot \sqrt{\tau} \geq \epsilon^* - \sigma \cdot \sqrt{\tau}) \\ &= P(-\epsilon + \sigma \cdot \sqrt{\tau} \leq -\epsilon^* + \sigma \cdot \sqrt{\tau}) \\ &= \mathcal{N}(-\epsilon + \sigma \cdot \sqrt{\tau}) \end{aligned}$$

– We will now propose the following new notation:

$$\begin{cases} d_1 = -\epsilon^* + \sigma \cdot \sqrt{\tau} \\ d_2 = -\epsilon^* \end{cases}$$

– With this new notation, the expression for the expectation of the value of the European-style long call option can be rewritten. Mathematically, this becomes:

$$E(f_T^{Eur.\ long\ call}) = S_{t_0} \cdot e^{r\cdot\tau} \cdot \mathcal{N}(d_1) - K \cdot \mathcal{N}(d_2)$$

- With this, we derived an expression for the expectation of the value of the payoff of the European-style long call option at maturity. At his time we can calculate the present value of this expected payoff by discounting this expectation. Mathematically, this becomes:

$$\begin{aligned} f_{t_0}^{Eur.\ long\ call} &= e^{-r\cdot\tau}\cdot E(f_T^{Eur.\ long\ call}) \\ &= e^{-r\cdot\tau}\cdot\Big[S_{t_0}\cdot e^{r\cdot\tau}\cdot\mathcal{N}(d_1) - K\cdot\mathcal{N}(d_2)\Big] \\ &= S_{t_0}\cdot\mathcal{N}(D_1) - K\cdot e^{-r\cdot\tau}\cdot\mathcal{N}(d_2) \end{aligned}$$

  The first term is equal to the present value of the expected benefits. The second term reflects the present value of the expected exercise cost. $\mathcal{N}(d_1)$ reflects the probability to be in the money in the risk-neutral world.

- In the previous paragraphs, we derived an expression for the Black-and-Scholes-Merton option pricing model. We now restate the mathematical expression of this model:

$$f_{t_0}^{Eur.long\ call} = S_{t_0}\cdot\mathcal{N}(d_1) - K\cdot e^{-r\cdot\tau}\cdot\mathcal{N}(d_2)$$

  Where d_1 and d_2 were defined in the following manner:

$$\begin{cases} d_1 &= \frac{ln(\frac{S_{t_0}}{K})+(r+0.5\cdot\sigma^2)\cdot\tau}{\sigma\cdot\sqrt{\tau}} \\ d_2 &= d_1 - \sigma\cdot\sqrt{\tau} \end{cases}$$

– Notice that, for an option to be very deep in the money:

1. $\mathcal{N}(d_2)$ would converge to one.
2. $\mathcal{N}(d_1)$ would converge to one.

If that were the case, the value of the European-style long call option would simplify to the value of the corresponding long forward contract. Mathematically:

$$\begin{aligned} f_{t_0}^{Eur.\ long\ call} &= S_{t_0} - K \cdot e^{-r \cdot \tau} \\ &= f_{t_0}^{Long\ fw.} \end{aligned}$$

- We not need to derive a separate model for put option pricing while the put-call parity can be used to derive such a model. Mathematically, this becomes:

$$\begin{aligned} f_{t_0}^{Eur.\ long\ call} + K \cdot e^{-r \cdot \tau} &= f_{t_0}^{Eur.\ long\ put} + S_{t_0} \\ f_{t_0}^{Eur.\ long\ put} &= f_{t_0}^{Eur\ long\ call} + K \cdot e^{-r \cdot \tau} - S_{t_0} \\ &= S_{t_0} \cdot \mathcal{N}(d_1) - K \cdot e^{-r \cdot \tau} \cdot \mathcal{N}(d_2) + K \cdot e^{-r \cdot \tau} - S_{t_0} \\ &= -S_{t_0} \cdot \Big[1 - \mathcal{N}(d_1)\Big] + K \cdot e^{-r \cdot \tau} \cdot \Big[1 - \mathcal{N}(d_2)\Big] \\ &= K \cdot e^{-r \cdot \tau} \cdot \mathcal{N}(-d_2) - S_{t_0} \cdot \mathcal{N}(-d_1) \end{aligned}$$

14.4 The volatility

- In this section we will discuss the important matters concerning the volatility σ.
- First, recall the expression for the Black-and-Scholes-Merton option pricing model. However, in this case we make the important distinction between τ_1 which reflects the number of trading days within the given period and τ_2 which reflects the number of calendar days within the given period . Mathematically, this becomes:

$$f_{t_0}^{Eur.\ long\ call} = S_{t_0} \cdot \mathcal{N}(d_1) - K \cdot e^{-r \cdot \tau_1} \cdot \mathcal{N}(d_2)$$

Where d_1 and d_2 were defined in the following manner:

$$\begin{cases} d_1 &= \frac{ln(\frac{S_{t_0}}{K}) + (r + 0.5 \cdot \sigma^2) \cdot \tau_2}{\sigma \cdot \sqrt{\tau}} \\ d_2 &= d_1 - \sigma \cdot \sqrt{\tau} \end{cases}$$

- Notice that the volatility σ is generated during trading days. Id est the value of the underlying does not fluctuate on days other than trading days. For this reason, τ_1 is calculated on the basis of the number of trading days in a year which is equal to 252 days. Interest calculations are done on the basis of the number of calendar days in a year. For this reason, τ_2 is calculated on the basis of the number of calendar days in the year. In summary, we use calendar days for discounting and trading days for rescaling the volatility.
- In general, there are two approaches to measuring the volatility σ.

1. First we have the unconditional volatility. The unconditional, historic volatility takes a window of past returns and calculates the standard deviation to get an estimate of the volatility over the next period. This is done, following the following steps.

   – First, we compute n continuous returns, on a daily basis. Mathematically:

   $$r = ln(\frac{s_t}{s_{t-1}})$$

   – Furthermore, we estimate the unbiased standard deviation of the n continuous returns. Mathematically:

   $$s = \sqrt{\frac{1}{n-1} \cdot \sum_{i=1}^{n}(r - \bar{r}^2)}$$

   – As we know already, the standard deviation s is an estimate of the population standard deviation σ. We can therefore compute the population standard deviation σ. Mathematically:

   $$\sigma = \frac{\hat{s}}{\sqrt{\tau}}$$

2. Secondly, we have the conditional volatility which comprises different methods to calculate the volatility. Some examples are given here.

   – GARCH models i.e. generalized autoregressive conditional heteroscedasticity. GARCH models are time series models in which we are making the volatility vary with time. The future volatility is a therefore function of the past volatility.

   – A second alternative is called implied volatility estimates. This method backs out market information to obtain volatility estimates. In this case, the volatility is an estimate over the lifetime of the option contract.

     This method is hampered by smiles and smirks in τ and K. Id est options with different time to maturities and strike prices have different volatilities. Options that are in the money have a different volatility than options that are out of the money. A solution consists of modeling a volatility surface where the x-axis represents the time to maturity, where the y-axis represents the moneyness of the option as measured by $\frac{S}{K}$ and where the z-axis represents the implied volatility. We can then use weighting schemes where a greater weight is given to options that are more sensitive to volatility changes i.e options that are in the money.

   – A third alternative involves using a fear index such as the Vix. The Vix is an index of implied volatilities, computed from 30-day S&P500 options. A quote of 15 represents a volatility of 15% p.a. The Vix future is a future contract where the underlying is the Vix index and where the future price is the Vix-index times one thousand.

14.5 The dividend cases

- Dividends play no role in the case of European-style options. In the case of American-style options where there is no dividend, both the Black-and-Scholes model and the binomial option pricing model can be used without any adaptations.
- Dividends can be modeled in a continuous manner, using a dividend yield γ and in a discrete way, expressing the value of each dividend D_i, separately.
- Consider a long position in a future contract on an underlying that provides a dividend during its lifetime. The value of that future contract is adjusted by subtracting the present value of the dividend. Mathematically, this becomes:

$$f_{t_0}^{Long\ fut.} = S_T \cdot e^{-\gamma \cdot \tau} - K$$
$$f_{t_0}^{Long\ fut.} = S_T - PV(D) - K$$

- Consider a European call option on an underlying that does not provide a dividend.
  - The Black-and-Scholes formula computes the price of this option in the following manner:

$$f_{t_0}^{Eur.\ long\ call} = S_{t_0} \cdot \mathcal{N}(d_1) - K \cdot e^{(-r \cdot \tau)} \cdot \mathcal{N}(d_2)$$

    Where d_1 and d_2 were defined in the following manner:

$$\begin{cases} d_1 &= \frac{ln(\frac{S_{t_0}}{K}) + (r + \frac{\sigma^2}{2}) \cdot \tau}{\sigma \cdot \sqrt{\tau}} \\ d_2 &= d_1 - \sigma \cdot \sqrt{\tau} \end{cases}$$

  - We suggest the following adaptation for the case where the asset, underlying the option contract provides a dividend yield γ.

$$f_{t_0}^{Eur.\ long\ call} = S_{t_0} \cdot e^{-\gamma \cdot \tau} \cdot \mathcal{N}(d_1) - K \cdot e^{-r \cdot \tau} \cdot \mathcal{N}(d_2)$$

    Where d_1 and d_2 are defined in the following manner:

$$\begin{cases} d_1 &= \frac{ln(\frac{S_{t_0} \cdot e^{-\gamma \cdot \tau}}{K}) + (r + \frac{\sigma^2}{2}) \cdot \tau}{\sigma \cdot \sqrt{\tau}} \\ d_2 &= d_1 - \sigma \cdot \sqrt{\tau} \end{cases}$$

    Or alternatively:

$$\begin{cases} d_1 &= \frac{ln(\frac{S_{t_0}}{K}) + (r - \gamma + \frac{\sigma^2}{2}) \cdot \tau}{\sigma \cdot \sqrt{\tau}} \\ d_2 &= d_1 - \sigma \cdot \sqrt{\tau} \end{cases}$$

- The expression for the European-style long put option, in the continuous dividend case, becomes:

$$f_{t_0}^{Eur.\ long\ put} = K \cdot e^{-r\cdot\tau} \cdot \mathcal{N}(-d_2) - S_{t_0} \cdot e^{-\gamma\cdot\tau} \cdot \mathcal{N}(-d_1)$$

Where d_1 and d_2 are defined in the following manner:

$$\begin{cases} d_1 &= \frac{ln(\frac{S_{t_0}\cdot e^{-\gamma\cdot\tau}}{K})+(r+\frac{\sigma^2}{2})\cdot\tau}{\sigma\cdot\sqrt{\tau}} \\ d_2 &= d_1 - \sigma \cdot \sqrt{\tau} \end{cases}$$

Or alternatively:

$$\begin{cases} d_1 &= \frac{ln(\frac{S_{t_0}}{K})+(r-\gamma+\frac{\sigma^2}{2})\cdot\tau}{\sigma\cdot\sqrt{\tau}} \\ d_2 &= d_1 - \sigma \cdot \sqrt{\tau} \end{cases}$$

- The Black-and-Scholes model is intended for European-style options. In the case of of American-style call options, where the underlying asset does not provide discrete dividends during the lifetime of the option, the option price will be equal to the option price of the corresponding European-style long call option. In the case of American-style call options, where the underlying asset provides discrete dividends during the lifetime of the option, it cannot be precluded that the option will be exercised cum-dividend. In this case, the option price can be determined by using the Roll-Geske-Whayley analytical solution or by using Black's approximation which is easier to implement. Neither will be discussed.

- We summarize our findings.

  – in the case where the asset underlying the option provides dividends, the value of the underlying at maturity S_T in the Black-and-Scholes model is replaced with the value of the underlying at maturity, discounted with a dividend yield γ or subtracted by the sum of the present value of each individual dividend that is provided by the underlying during the lifetime of the option contract. Mathematically:

$$f_{t_0}^{Eur.\ long\ call} = S_{t_0} \cdot e^{-\gamma\cdot\tau} \cdot \mathcal{N}(d_1) - K \cdot e^{-r\cdot\tau} \cdot \mathcal{N}(d_2)$$
$$f_{t_0}^{Eur.\ long\ call} = \Big[S_{t_0} - PV(D)\Big] \cdot \mathcal{N}(d_1) - K \cdot e^{-r\cdot\tau} \cdot \mathcal{N}(d_2)$$

  – By including a dividend, the level of the process is lowered. We should therefore also adjust the volatility: $\frac{S_{t_0}}{S_{t_0}-D} \cdot \sigma$ The volatility will be somewhat higher. Id est we have the same absolute shocks, but we apply them on a lower level.

14.6 Currency options

- We now take a look at currency options. More specifically, we will determine the price of a European-style long call currency option.
  - Recall the expression for the Black-and-Scholes-Merton option pricing model. Mathematically:

$$f_{t_0}^{Eur.\ long\ call} = S_{t_0} \cdot e^{-\gamma \cdot \tau} \cdot \mathcal{N}(d_1) - K \cdot e^{-r \cdot \tau} \cdot \mathcal{N}(d_2)$$

  - In this case the dividend yield γ should be interpreted as the interest rate provided by the foreign currency. Mathematically, this becomes:

$$\begin{aligned} f_{t_0}^{Eur.\ long\ call} &= S_{t_0} \cdot e^{-\gamma \cdot \tau} \cdot \mathcal{N}(d_1) - K \cdot e^{-r \cdot \tau} \cdot \mathcal{N}(d_2) \\ &\Big| \gamma = r_{for.} \\ &= e^{r \cdot \tau} \cdot e^{-r \cdot \tau} \cdot S_{t_0} \cdot e^{r_{for.} \cdot \tau} \cdot \mathcal{N}(d_1) - K \cdot e^{-r \cdot \tau} \cdot \mathcal{N}(d_2) \\ &= S_{t_0} \cdot e^{(r - r_{for.}) \cdot \tau} \cdot e^{-r \cdot \tau} \cdot \mathcal{N}(d_1) - K \cdot e^{-r \cdot \tau} \cdot \mathcal{N}(d_2) \\ &= \Big[f_t^{Long\ fw.} \cdot \mathcal{N}(d_1) - K \cdot \mathcal{N}(d_2) \Big] \cdot e^{-r \cdot \tau} \end{aligned}$$

  Where d_1 and d_2 are defined in the following manner:

$$\begin{cases} d_1 &= \frac{ln(\frac{S_{t_0} \cdot e^{-r_{for.} \cdot \tau}}{K} + (r + \frac{\sigma^2}{2} \cdot \tau))}{\sigma \cdot \sqrt{\tau}} \\ &= \frac{ln(\frac{S_{t_0} \cdot e^{(r - r_{for.}) \cdot \tau} \cdot e^{-r \cdot \tau}}{K} + (r + \frac{\sigma^2}{2} \cdot \tau))}{\sigma \cdot \sqrt{\tau}} \\ &= \frac{ln(\frac{f_{t_0}^{Long\ fw.} - r \cdot \tau + r \cdot \tau + \frac{\sigma^2}{2} \cdot \tau}{K})}{\sigma \cdot \sqrt{\tau}} \\ &= \frac{ln(\frac{f_{t_0}^{Long\ fw.} + \frac{\sigma^2}{2} \cdot \tau}{\sigma \cdot \sqrt{\tau}})}{den} \\ d_2 &= d_1 - \sigma \cdot \sqrt{\tau} \end{cases}$$

- This model can be used in different applications:
  - This model can be used to model exchange rates, in such a case, the dividend yield is a continuous yield.
  - The model can also be used to model an option on an index. We then express many small dividend payments as a yield over the maturity of the option. The dividend yield is thus an average dividend yield over the lifetime of the option.

15 Delta hedging with Black-Scholes-Merton

15.1 Introductory example

- In this chapter, we will investigate how we can use options to convert a portfolio into a riskless portfolio i.e. how we can use option for hedging purposes.
- In this regard, it is important to understand the sensitivity of the value of an options with respect to the underlying. These sensitivities are called the Greeks. Studying these sensitivities will allow us to better hedge a position with options.
- First, we will illustrate how the Black-and-Scholes-Merton option pricing model can be used to set up a hedge using an introductory example.
  - If a financial institution sells a derivative it makes a profit because there is a bid-ask spread. The financial institution is not interested making a profit by speculation. Id est the financial institution is not interested in creating an exposure. The financial institution will therefore attempt to hedge the exposure, after selling the derivative.
  - Suppose the financial institution was able to sell 100.000 European call options on an underlying that provides no dividends during the lifetime of the option, for a price \$300.000. The current price of the stock is \$49. The strike price was set to \$50. The risk-free interest rate is equal to 5%. The time to maturity is equal to 20 weeks. The volatility is equal to 20% p.a.
  - According to the black and Scholes model, the price of the call option when it was sold is equal to \$2.4. The total cost of all options combined, at the time of the sale, is therefore equal to $\$2.4 \cdot 100.000 = \$240.000$.
  - The value of this position at the time of the sale can be computed as the difference between the total amount of premia received and the total value of all options. Mathematically, this becomes:

$$\$300.000 - \$240.000 = \$60.000$$

– At this point in time, the financial institution can undertake one of several possible actions.

1. The financial institution could decide to keep the naked position on the books.

   If the price of the underlying asset at maturity is less then \$50, the payoff of the option is equal to zero. This would results in a profit of \$300.000. If the price of the underlying asset at maturity is greater than \$50, the payoff of the option is negative. An increase of the option price above \$50.6, would result in a loss for the financial institution. Mathematically, this becomes:

$$\begin{cases} S_T < 50 \rightarrow f_T^{Portf.} & = 0 \\ S_T > 50 \rightarrow f_T^{Portf.} & = -100.000 \cdot (S_T - 50) \end{cases}$$

2. The financial institution could also opt to use a covered call option strategy by purchasing 100.000 stocks upfront. The payoff of this portfolio per unit of the underlying, is given by:

$$\begin{aligned} f_T^{Portf.} &= (S_T - \$49) - max\ (S_T - \$50, 0) \\ &= min\ (\$1, S_T - \$49) \end{aligned}$$

   If the price of the underlying decreases below $48, 40$, this will result in a loss for the company.

– Both strategies are not very good. We observe that the naked position works out well if the stock price decreases below the strike price while the covered call strategy works out well if the stock price rises above the strike price. The question is whether we could combine the best of both worlds. In theory, we could try to set up a stop-loss strategy. This corresponds to taking the covered call position if $S_T > K$ while taking the naked position if $S_T < K$.

– However, this strategy is difficult to implement in practice. More specifically, the cashflows resulting from this strategy would occur at random times. It is therefore likely that the present value of these cashflows will not be equal to zero. Furthermore, trading will incur transaction costs which could quickly sky-rocket. Finally, it would be difficult to time the transactions exactly.

– For these reasons, the financial institution will have to buy and sell at prices that deviate from the actual strike price. Mathematically, this becomes:

$$\begin{cases} \text{Buy at } K + \Delta \\ \text{Sell at } K - \Delta \end{cases}$$

– These observations result in a *catch-22* situation. If Δ is kept small, we potentially have to trade a lot. If Δ is large, we deviate from the hedging idea.

15.2 Dynamic delta hedging

- In the previous section we discussed hedging an option position. This was done using a stop-loss strategy. This meant we by bought and sold the number of units of the underlying equal to the number of options, each time the price of the underlying would cross certain boundaries. Id est we bought or sold the whole block of the underlying each time the price of the underlying would cross a certain boundary. This strategy was difficult to implement in practice.
- We could improve upon this strategy by setting up a replicating portfolio for the option portfolio and by continuously adapting this portfolio. Such a strategy was called dynamic delta hedging.
- Recall that a riskless position could be obtained, by combining one short European call option with Δ units of the underlying asset. The Delta Δ was the rate of change in the option price, with respect to the price of the underlying. The Delta measured the slope in the relationship between the option and the value of the underlying asset. For example, a Delta of $+0.6$ means that a small increase in the price of the underlying asset of Δ will result in an increase of $0.6 \cdot \Delta$ in the option price. Mathematically:

$$\Delta = \frac{\delta f}{\delta S}$$

- We will now derive the expression for the delta of various instruments. Computing the delta boils down to calculating the first derivative.
  - Delta of a long call option.

$$0 \leq \frac{\delta f}{\delta S} \leq 1$$

  - Delta of a long put option.

$$-1 \leq \frac{\delta f}{\delta S} \leq 0$$

  - Delta of the underlying.

$$\frac{\delta S}{\delta S} = 1$$

  - Delta of a long forward contract, on a non-dividend paying asset.

$$\frac{\delta f}{\delta S} = \frac{\delta(S - K \cdot e^{-r \cdot \tau})}{\delta S} = 1$$

  - Delta of a long forward on a dividend paying asset.

$$\begin{aligned}\frac{\delta f}{\delta S} &= \frac{\delta(S \cdot e^{-\gamma \cdot T}) \cdot e^{-r \cdot \tau}}{\delta S} \\ &= e^{-\gamma \cdot \tau}\end{aligned}$$

- We will now show how a delta hedge can be constructed by considering the following example.
  - Suppose we have 20 short call options and that the contract size is equal to 100. The Delta for a long call is equal to 0.6. If the price of the underlying increases with one dollar, the price of one call option will increase with sixty cents.
  - The goal here is to make the portfolio immune to changes in the price of the underlying. This portfolio can be hedged using options or using the underlying itself.
  - We first consider the latter method where we add a number of units of the underlying to the portfolio such that the delta of the options is exactly compensated. The delta of the portfolio after entering into the long position in the underlying is the sum of both delta's. If we add the underlying to the portfolio, the total delta of the portfolio would be equal to zero and the portfolio would therefore become delta neutral. Mathematically:

$$\begin{aligned} \Delta_{\,Portf.} &= \Delta_{\,Short\ call\ portf.} + \Delta_{\,Long\ spot\ portf.} \\ &= 0 \end{aligned}$$

  - The question now becomes, what information we need to calculate the delta of a given position in a given instrument. In our example, we can compute the delta of the short call portfolio in the following manner:

$$\begin{aligned} \Delta_{\,Short\ call\ portf.} &= -(20 \cdot 100) \cdot 0.6 \\ &= -1200 \end{aligned}$$

  - This computation comprises the following elements:
    1. The direction of the relationship between the price of the derivative instrument and the price of the underlying asset.
    2. The size of the derivatives position. Id est the number of derivative instruments within the portfolio that is to be hedged.
    3. The sensitivity of the derivative with regard to the underlying.
  - The portfolio can be hedged by entering into a long spot position, of 1200 units, in the underlying. Mathematically:

$$\begin{aligned} \Delta_{\,Portf.} &= \Delta_{\,Short\ call\ portf.} + \Delta_{\,Long\ spot\ portf.} \\ 0 &= -1200 + n \cdot \Delta_S \\ n &= 1200 \end{aligned}$$

  - Recall that when the option is deep in the money, the delta Δ will be equal to one and when the option is deep out of the money, the delta Δ will be equal to zero. Id est, the delta changes whenever the price of the underlying changes. This means we will have to adjust the hedge, during the lifetime of the hedge, based on the delta of the option. This called dynamic hedging.

- Consider the following scenario where the stock price increases in value and causes the delta of the call option to increase from 0.6 to 0.7. The portfolio will now have a net delta position of −200.

$$\begin{cases} \Delta_{\text{ Short call position}} & = -2000 \cdot 0.7 = -1400 \\ \Delta_{\text{ Long spot position}} & = +1200 \cdot 1 = +1200 \\ \Delta_{\text{ Portf.}} & = -200 \end{cases}$$

- It is clear that, in order to become delta-neutral again, the financial institution will have to purchase 200 additional units of the underlying. Such a procedure is known as rebalancing.

- Consider a new example.
  - We are given a portfolio that contains the following components:
    1. 100 long call options with a delta of 0.6.
    2. 200 short call options with a delta of 0.4.
    3. 50 short put options with a delta of −0.30.
  - The contract size of an option contract is equal to 100 units of the underlying. This portfolio can be made delta-neutral by going long in 500 units of the underlying. We derived this by calculating the delta of the portfolio.

$$\begin{aligned} \Delta_{Portf.} &= \Delta_{\text{ Long call pos.}} + \Delta_{\text{ Short call pos.}} + \Delta_{\text{ Short put pos.}} \\ &= +10.000 \cdot 0.6 - 20.000 \cdot 0.4 - 50.000 \cdot -0.30 \\ &= -500 \end{aligned}$$

  - In general, the delta of a portfolio of derivatives, written on a single underlying asset, is given by the weighted average of the deltas of the individual derivatives. Mathematically, this becomes:

$$\Delta_{portfolio} = \sum_{i=1}^{n} w_i \cdot \Delta_i$$

Where Δ_i is the delta of the i^{th} option and w_i is the number of options held.

- The question now rises whether we need the underlying itself to perform a delta-hedge. The answer to this question is negative.
  - Consider for example a portfolio of currency options, held by a US-bank which could be made delta neutral by going short in the underlying for £458.000. The risk-free interest rate is equal to 4% in the United States and 7% in the United Kingdom.
  - The delta of a short position in a futures contract was given by the following expression:

$$\Delta_{\;Short\;fut.} = e^{-(r-r_{for.})\cdot\tau}$$

  - The delta of the entire position in the short futures, is computed by multiplying the delta of short future contract with the size of the position. Mathematically:

$$\begin{aligned}\Delta_{Portf.} &= \Delta_{\;Short\;fut.} \cdot n \\ &= e^{-(0.04-0.07)\cdot\frac{9}{12}} \cdot \$458.000 \\ &= \$468.422\end{aligned}$$

  - A currency future has a contract size of £62.500. We would therefore need to enter seven short forward contract to hedge this position. Mathematically:

$$\begin{aligned}n &= \frac{\$468.442}{\$62.500} \\ &= 7.50\end{aligned}$$

  - In conclusion, the hedge ratio based on currency futures H_F, is given by:

$$H_F = e^{-(r-r_{for.})\cdot\tau} \cdot H_A$$

    Where H_A is the delta of the position we need to hedge. If we would put up a hedge with the underlying itself, this would be equal to one.

16 The Greeks

- Portfolio traders not only like to hedge the sensitivity of the options within the portfolio vis à vis the price of the underlying but also is à vis other variables. For example, a portfolio trader would likely want the delta of his portfolio to remain relatively constant over time. The portfolio trader would thus like the sensitivity of the delta itself to be small. The sensitivity of the delta corresponds to the second derivative of the price of the option with regard to the price of the underlying. This sensitivity is called the gamma. We will also consider the sensitivity of the options within the portfolio vis à vis other variables such as interest rates, the time to maturity and the volatility.

16.1 The Delta

- The delta of a long call option corresponds to the parameter $\mathcal{N}(d_1)$ in the Black-and-Scholes-Merton model. The delta of a long put option corresponds to $\mathcal{N}(d_1) - 1$. In the case where there is a dividend, the delta of the option is discounted with the dividend yield γ. Mathematically:

$$\begin{aligned}\Delta_{\ \textit{Long call ndiv}} &= \mathcal{N}(d_1)\\ \Delta_{\ \textit{Long put ndiv}} &= \mathcal{N}(d_1)\\ \Delta_{\ \textit{Long call div}} &= \mathcal{N}(d_1) \cdot e^{-\gamma \cdot \tau}\\ \Delta_{\ \textit{Long put div}} &= \mathcal{N}(d_1) \cdot e^{-\gamma \cdot \tau}\end{aligned}$$

- We can also compute the delta in a numerical way. This is done by considering the general expression of the first derivative of a function. We investigate the evolution of the value of the the derivative for small changes in the price of the underlying asset.

$$\begin{aligned}f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\\ f'(S) &= \lim_{\Delta S \to 0} \frac{f(S + \Delta S) - f(S)}{\Delta S}\end{aligned}$$

16.2 The Gamma

- The gamma Γ reflects the sensitivity of the delta Δ with respect to the price of the underlying asset S. The gamma therefore corresponds to the second derivative of the value of the derivative with regard to the price of the underlying asset. Mathematically, this becomes:

$$\begin{aligned}\Gamma_{\ Portf.} &= \frac{\Delta^2 \Pi}{\Delta S^2} \\ &= \frac{\Delta}{\delta S}\end{aligned}$$

- We can now easily calculate the gamma Γ for various types of derivatives. Consider for example the long call and the long put option. Mathematically, this becomes:

$$\Gamma_{\ Long\ call} = \Gamma_{\ Long\ put} = \frac{\mathcal{N}'(d_1) \cdot e^{-\gamma \cdot \tau}}{S \cdot \sigma \cdot \sqrt{\tau}}$$

  Where $\mathcal{N}'(d_1)$ can be computed, using the following equation:

$$\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi} \cdot e^{\frac{1}{2} \cdot x^2}}$$

- We will now try to create a portfolio that is not only delta-neutral but also gamma-neutral. This would imply the delta of the portfolio will be relatively insensitive to changes in the price of the underlying. A portfolio trader would therefore not need to rebalance his portfolio as much as before i.e. the hedge will not have to be adjusted as much.
- Notice that the gamma of the underling itself is always equal to zero. This means we cannot change the gamma of a portfolio of options by going long or short spot in the underlying. We will have to gamma-hedge by adding other options to the portfolio. After a portfolio is made gamma-neutral, we can easily change the delta by going long or short in the underlying, this will not change the gamma. In summary, we can make a portfolio delta-neutral by taking a position in the underlying, doing so will not affect the gamma of the portfolio. Mathematically:

$$\begin{aligned}\frac{\delta^2 S}{\delta S^2} &= \frac{\delta}{\delta S}\left(\frac{\delta S}{\delta S}\right) \\ &= \frac{\delta}{\delta S}\left(1\right) \\ &= 0\end{aligned}$$

- The delta is most sensitive to changes in the price of the underlying when an option is at the money. Id est the gamma is maximal when an option is at the money. When the option is deep in the money or deep out of the money, the delta is not sensitive to price changes.

16.3 The relationship between the delta and the gamma

- Recall the fundamental partial differential equation. Mathematically:

$$\Pi_T + r \cdot S \cdot \Pi_S + 0.5 \cdot \sigma^2 \cdot \Pi_{SS} = r \cdot \Pi$$

- Notice that this pde already told us that, in order to get into a riskless position i.e. to get the riskless rate r on the value of the portfolio Π, we have to take into account the derivative of the portfolio vis à vis S, the derivative of the portfolio vis à vis S^2 and the derivative of the portfolio vis à vis T. These derivatives in the fundamental pde correspond to the Greeks. Mathematically:

$$\begin{aligned} \Pi_S &= \Delta \\ \Pi_{SS} &= \Gamma \\ \Pi_T &= \Theta \end{aligned}$$

- For a delta-neutral portfolio, the theta depends on the value of the gamma. Because the gamma of individual options is always positive, the theta will be negative. This is because the time value of the option decreases with time. Mathematically:

$$\Theta = -0.5 \cdot \sigma^2 \cdot \Gamma + r \cdot \Pi$$

16.4 The Theta

- As we already know, the theta measures the rate of change of the value of the portfolio with respect to time. This change with time is negative and corresponds to the declining time value of the option with time.
- We can derive the expressions fo the theta for the long call and the long put option. By default, Θ is measured in years, by dividing the Θ by 252, we obtain daily changes. Mathematically:

$$\begin{aligned} \Theta_{\textit{Long call}} &= \frac{S \cdot \mathcal{N}'(d_1) \cdot \sigma}{2 \cdot \sqrt{\tau}} - r \cdot K \cdot e^{-r \cdot \tau} \cdot \mathcal{N}(d_2) \\ \Theta_{\textit{Long put}} &= \frac{S \cdot \mathcal{N}'(d_1) \cdot \sigma}{2 \cdot \sqrt{\tau}} - r \cdot K \cdot e^{-r \cdot \tau} \cdot \mathcal{N}(-d_2) \end{aligned}$$

  Where $N'(x)$ can be computed in the following way:

$$\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi} \cdot e^{\frac{1}{2} \cdot x^2}}$$

16.5 The Vega

- This parameter is not present in the Black-and-Scholes-Merton model. This is because the Black-Scholes-Merton model assumes the the volatility is constant over the lifespan of the option. In reality however, the volatility is time-varying. The vega is defined as the first derivative of the value of the portfolio Π with respect to the volatility σ. Mathematically, this becomes:

$$\nu = \frac{\delta \Pi}{\delta \sigma}$$

- The vega of the underlying asset is equal to zero. This is because, from a risk-neutral perspective, the value of the underlying asset does not depend on the volatility of that asset. The vega of the portfolio ν_{Π} can therefore only be changed by adding options to the portfolio.
- The vega of the long call option and the long put option is given by the following expression:

$$\nu_{Long\ call} = \nu_{Long\ put} = S \cdot \sqrt{\tau} \cdot \mathcal{N}'(d_1)$$

- Assume that the vega of our portfolio is denoted by ν_{Π} and the vega of the option we add to the portfolio is denoted by ν_{Option}. In order to make the portfolio vega-neutral, we choose n in such a way that the following expression holds true:

$$\nu_{\Pi} + n \cdot \nu_{Option} = 0$$

16.6 Delta-Gamma-Vega Hedging

- The aim of a option-portfolio trader will be to make his portfolio delta-neutral, gamma-neutral and vega-neutral.
  - In order to make the portfolio delta-neutral the trader can add a position in the underlying asset to his portfolio.
  - In order to make the portfolio gamma-neutral of vega-neutral, the trader will have to add option positions to his portfolio.
  - The portfolio has to be made gamma-neutral and vega-neutral at the same time. This is only possible when the hedger uses two separate option contracts. This gives us a system of two equations in two unknown variables n_1 and n_2 which represent the size of the position in the first option contract and the size of the position in the second option contract. The first option contract is added to the portfolio in order to make the portfolio gamma-neutral. The second option contract is added to the portfolio, in order to make the portfolio vega-neutral. Mathematically:

$$\begin{cases} \Gamma_{\Pi} + n_1 \cdot \Gamma_1 + n_2 \cdot \Gamma_2 = 0 \\ \nu_{\Pi} + n_1 \cdot \nu_1 + n_2 \cdot \nu_2 = 0 \end{cases}$$

- The gamma-vega neutral portfolio can be made delta-neutral by adding a position in the underlying asset.

16.7 The Rho

- The Black-Scholes-Merton model assumes a constant riskless interest rate over the lifespan of the option. In reality however, interest rates vary with time. The rho is defined as the first derivative of the value of the portfolio Π with respect to the risk-free interest rate r. The rho measures the sensitivity of the value of the option portfolio with respect to a change in the interest rate. Mathematically, this becomes:

$$\rho = \frac{\delta \Pi}{\delta r}$$

- The rho of the long call option and the long put option are given by the following expressions:

$$\rho_{\ Long\ call} = K \cdot \tau \cdot e^{-r \cdot \tau} \cdot \mathcal{N}(d_2)$$
$$\rho_{\ Short\ call} = -K \cdot \tau \cdot e^{-r \cdot \tau} \cdot \mathcal{N}(-d_2)$$